Orthogonal Sets

Geometric Algorithms
Lecture 22

CAS CS 132

Practice Problem

$$\mathscr{B} = \left(\underset{\vec{v}_1}{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}} \right)$$

Determine $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathscr{B}}$

$$\vec{v}_1 \cdot \vec{v}_2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$$

$$= 0$$

Answer

$$\begin{bmatrix}2\\3\end{bmatrix}_{\mathscr{B}}$$

express $\begin{bmatrix}2\\3\end{bmatrix}$ in the given basis

$$\mathscr{B} = \left(\begin{bmatrix}\frac{\sqrt{2}}{2}\\[4pt] \frac{\sqrt{2}}{2}\end{bmatrix}, \begin{bmatrix}-\frac{\sqrt{2}}{2}\\[4pt] \frac{\sqrt{2}}{2}\end{bmatrix}\right)$$

$\vec{v}_1$, $\vec{v}_2$

$[\vec{v}]_{\mathscr{B}}$

$$\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$$

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 \iff \underline{\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \vec{v}}$$

OR $\Rightarrow$ $\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}^{-1}[\vec{v}]$

COB matrix

$$\left[\begin{array}{cc|c}\sqrt{2}/2 & -\sqrt{2}/2 & 2\\ \sqrt{2}/2 & \sqrt{2}/2 & 3\end{array}\right]$$

$$\left[\begin{array}{cc|c}\sqrt{2}/2 & -\sqrt{2}/2 & 2\\ 0 & \sqrt{2} & 1\end{array}\right]$$

$$\left[\begin{array}{cc|c}\sqrt{2}/2 & 0 & 5/2\\ 0 & \sqrt{2} & 1\end{array}\right] \Rightarrow \begin{array}{l} a_1 = 5/\sqrt{2}\\ a_2 = 1/\sqrt{2}\end{array}$$

$$\cancel{\vec{v} = 2\vec{v}_1 + 3\vec{v}_2 = 2\begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix} + 3\begin{bmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix} = \begin{bmatrix}2\sqrt{2}/2 - 3\sqrt{2}/2\\ 2\sqrt{2}/2 + 3\sqrt{2}/2\end{bmatrix} = \begin{bmatrix}-\sqrt{2}/2\\ 5\sqrt{2}/2\end{bmatrix}}$$

Objectives

1. Recap analytic geometry in R^n
2. Try to understand why it is useful to work with orthogonal vectors
3. Get a sense of how to compute orthogonal vectors
4. Start to connect orthogonality to matrices and linear transformations

Keywords

orthogonal

orthogonal set

orthogonal basis

orthogonal projection

orthogonal component

orthonormal

orthonormal set

orthonormal basis

orthonormal matrix

orthogonal matrix

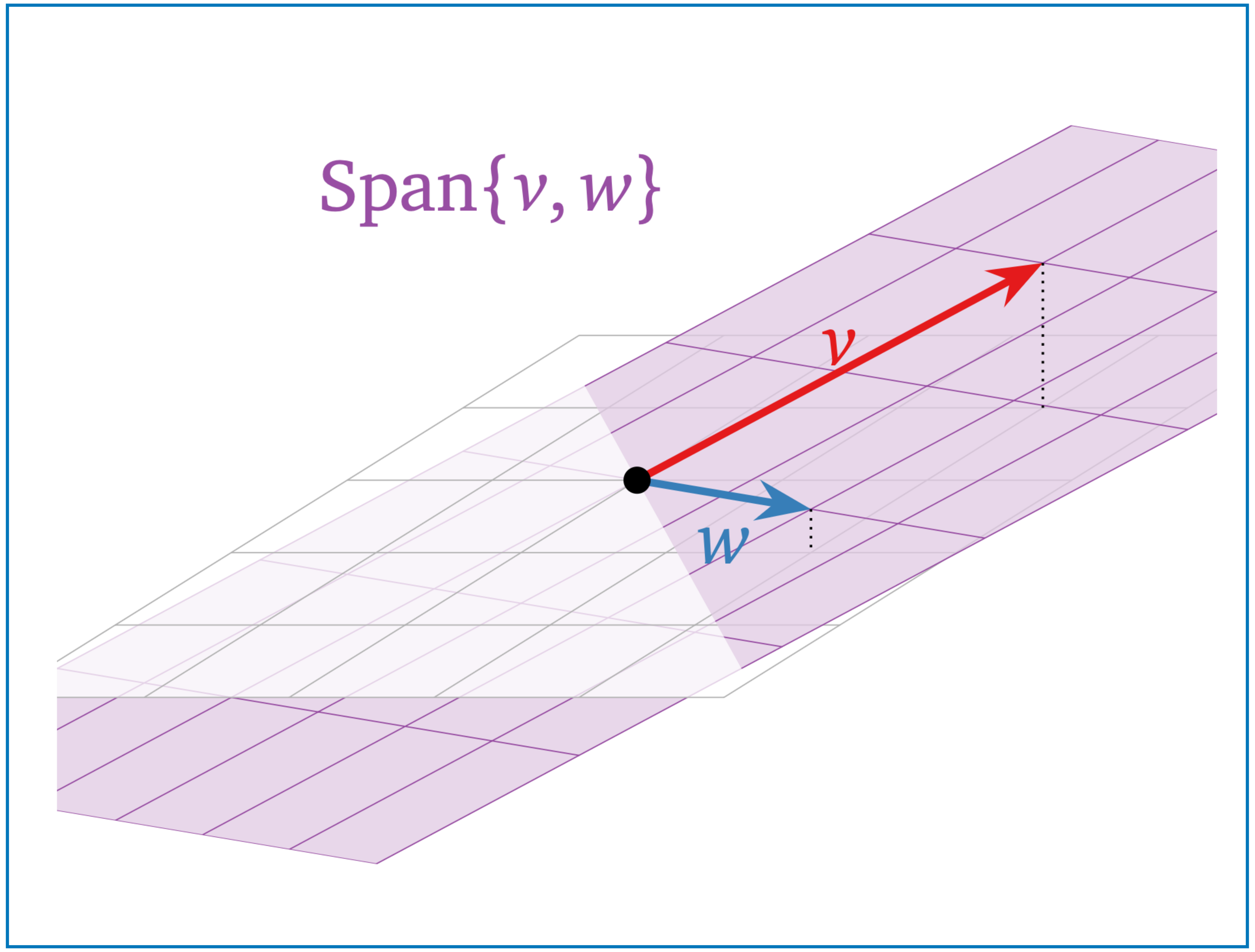
Recap: Analytic Geometry

Recall: The First Key Idea

Angles make sense in *any* dimension

Any pair of vectors in $\mathbb{R}^n$ span a (2D) plane

(We could formalize this via change of bases)



Recall: The Second Key Idea

All of the basic concepts of analytic geometry can be defined *in terms of inner products*

Spaces with inner products (like $\mathbb{R}^n$) are places where you can do analytic geometry

Recall: Inner Products

$$\underline{\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

matrix-vector multiplication

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Inner Products

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Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

a.k.a. dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Norms and Inner Products

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

The norm of a vector is the square root of the inner product with itself.

Recall: Norms and Inner Products

$$\sqrt{\sum_{i=1}^{n} v_i^2}$$

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

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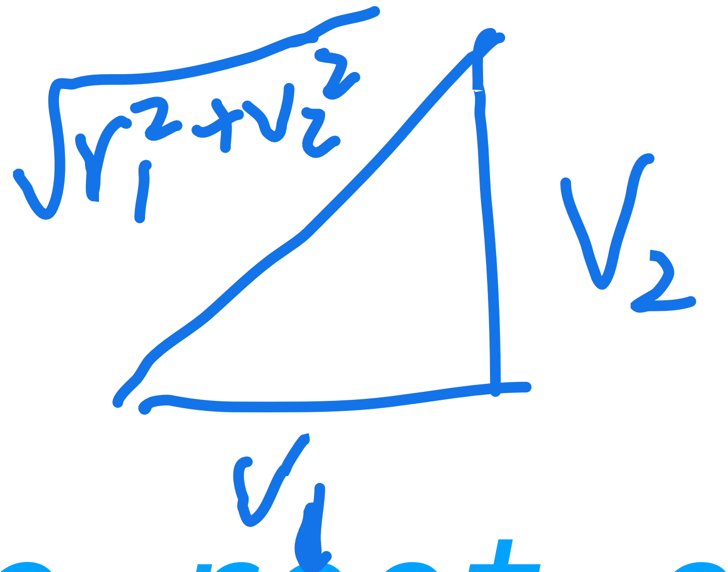


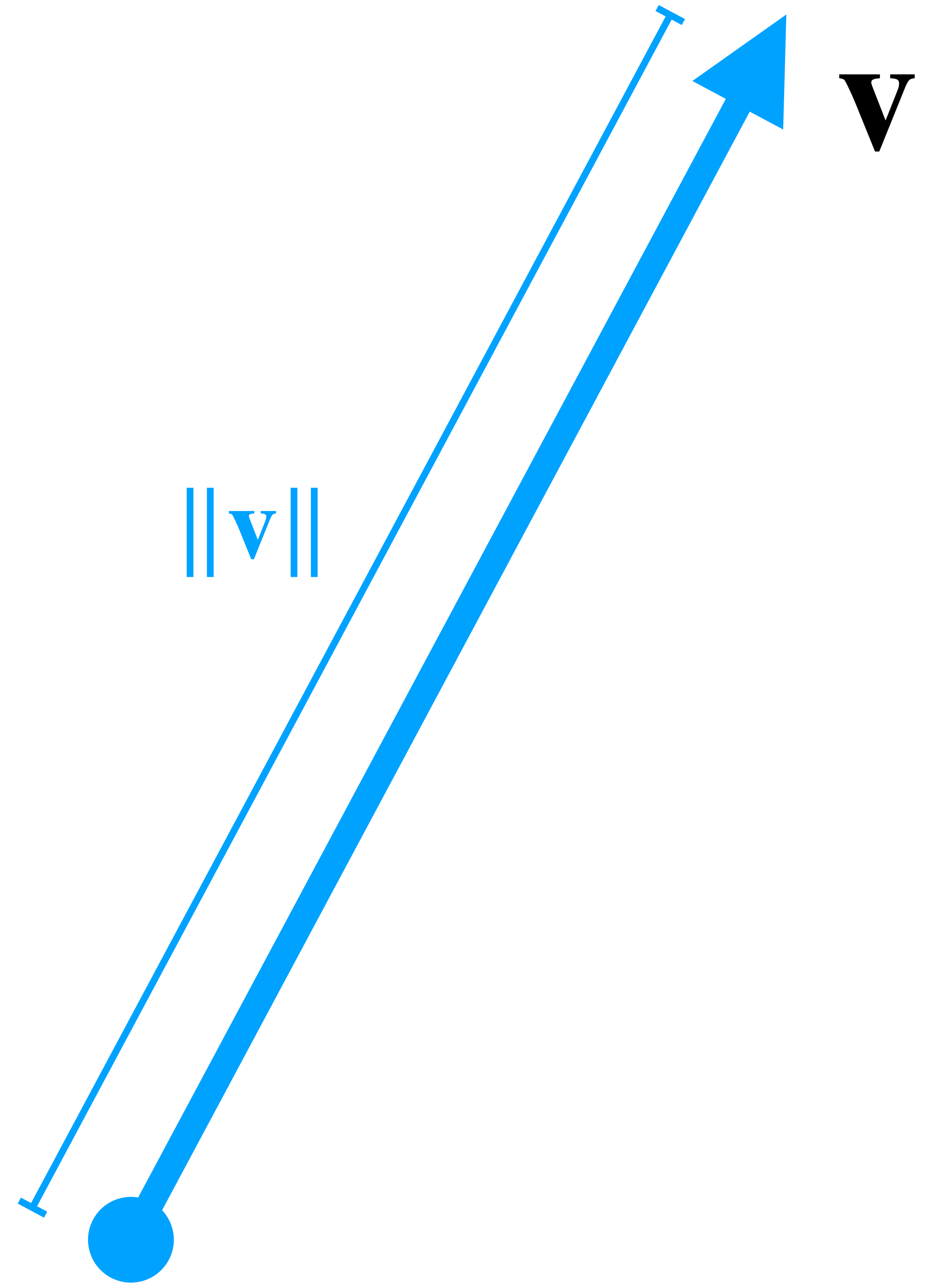
The norm of a vector is the square root of the inner product with itself.

It's important that $\mathbf{v}^T\mathbf{v}$ is nonnegative.

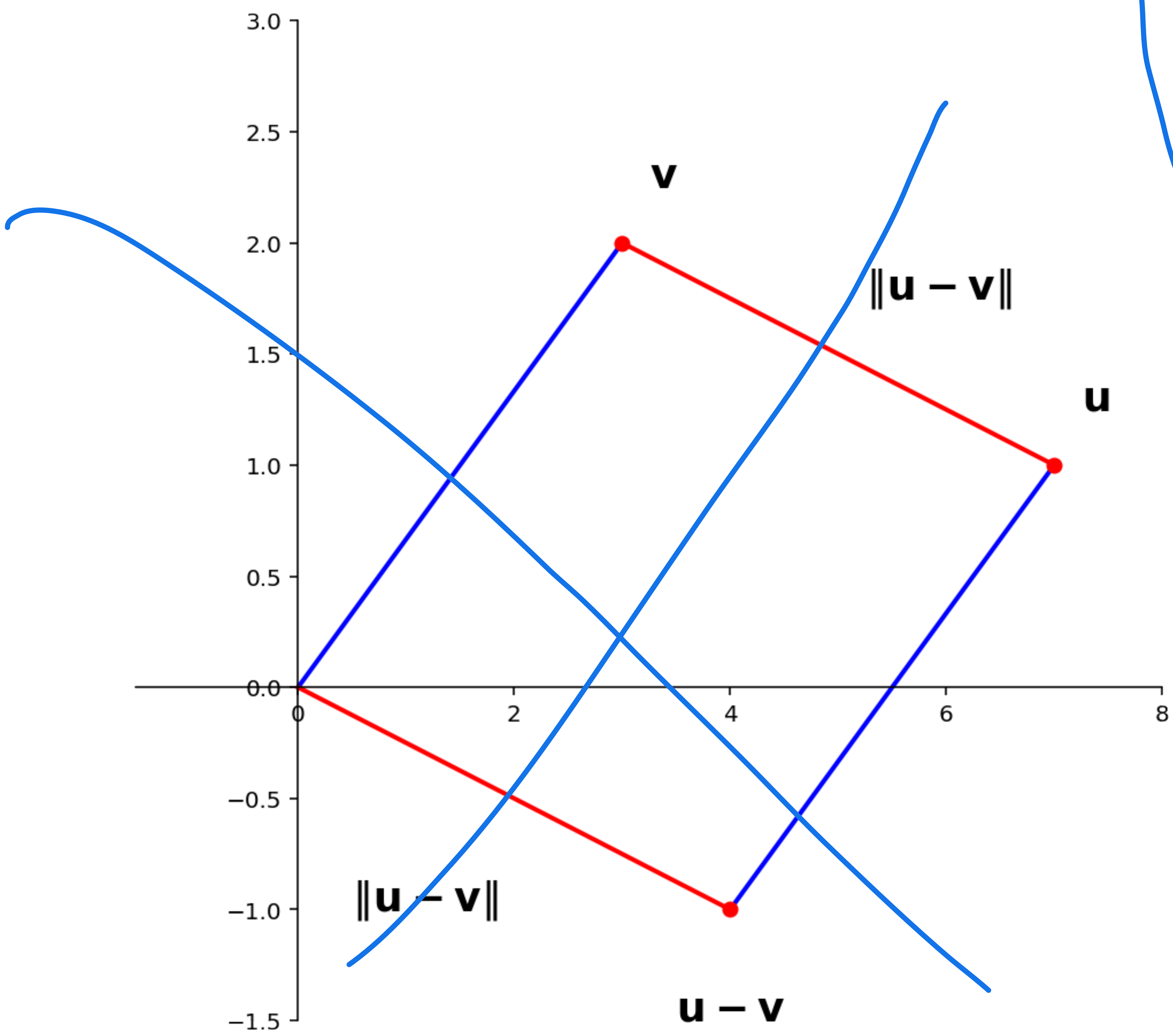
Recall: Norms and Length

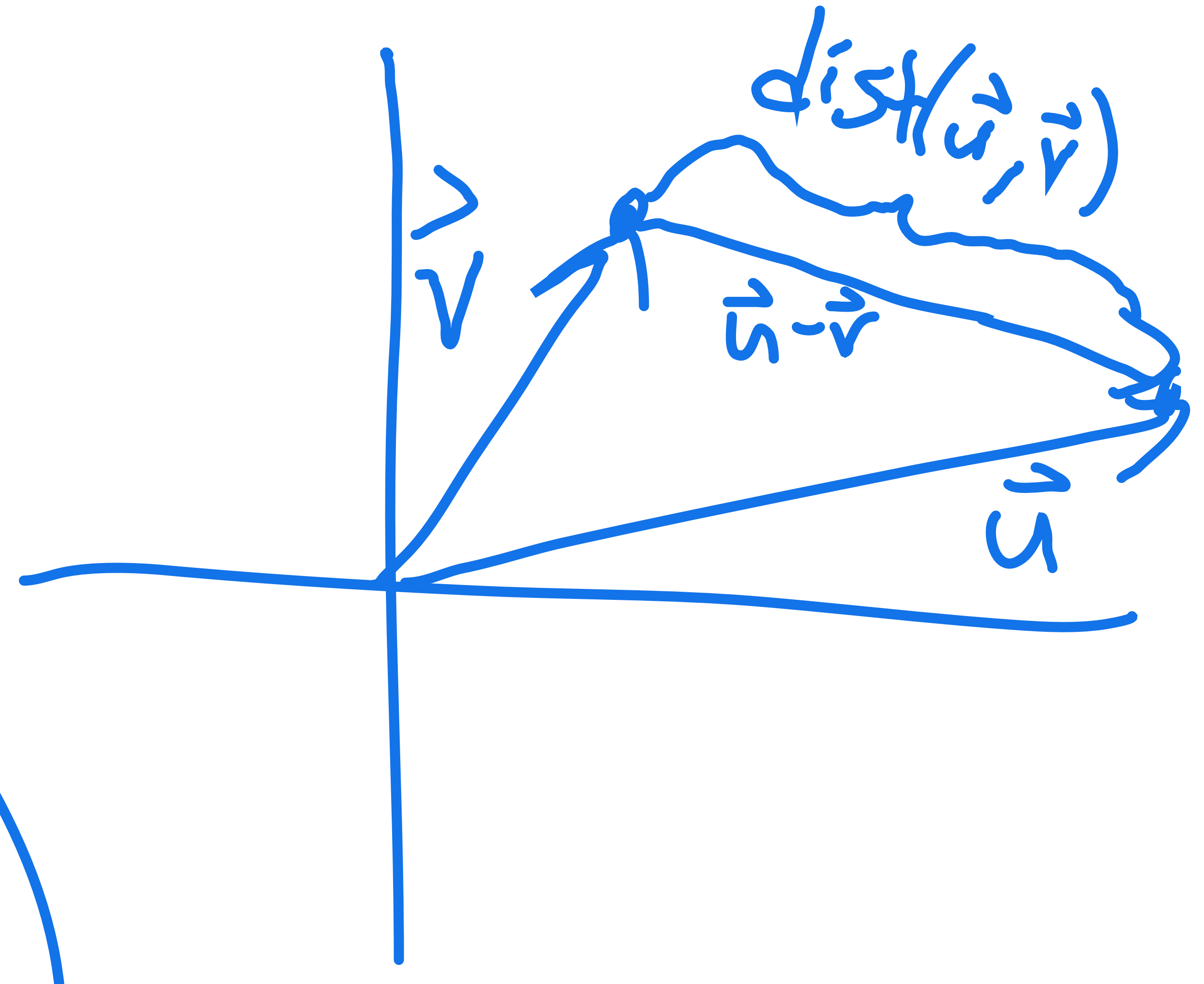
Norms give us a notion of <u>length</u>.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ this is our existing notion of length.



Recall: Distance (Pictorially)





$$\vec{v} + (\vec{u} - \vec{v}) = \vec{u}$$

$$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Recall: Distance (Algebraically)

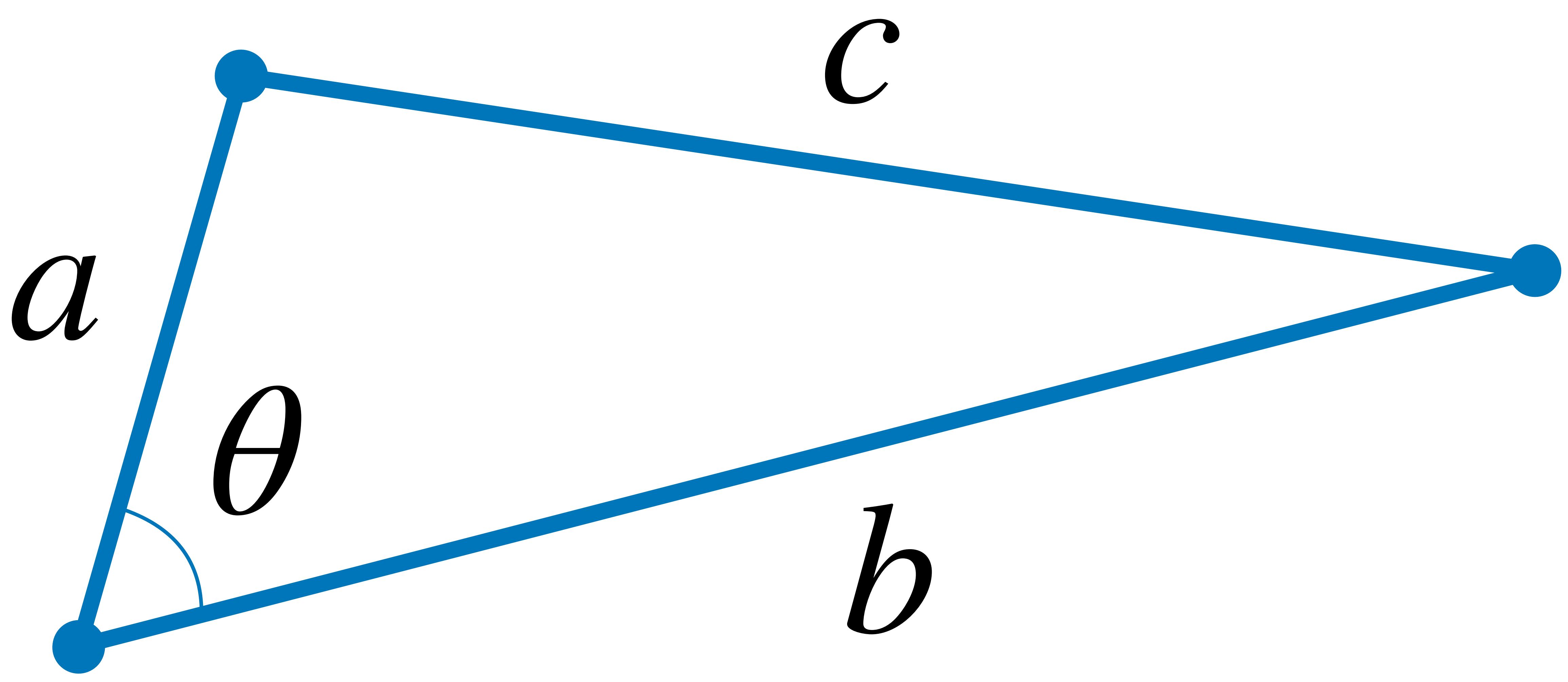
Definition. The distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is given by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g., $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\vec{u} - \vec{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $\quad \|\vec{u} - \vec{v}\| = \sqrt{16+1} = \sqrt{17}$

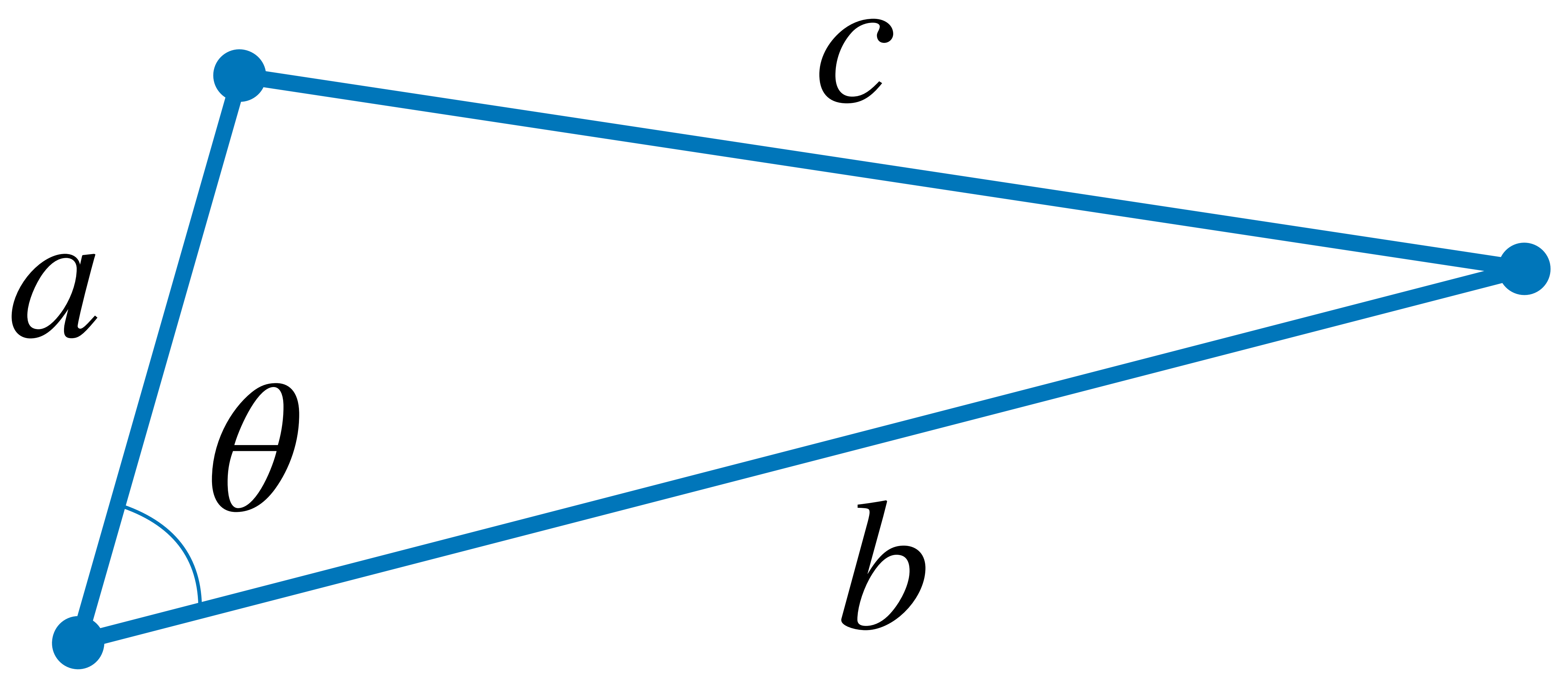
Recall: Law of Cosines



Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Recall: Law of Cosines

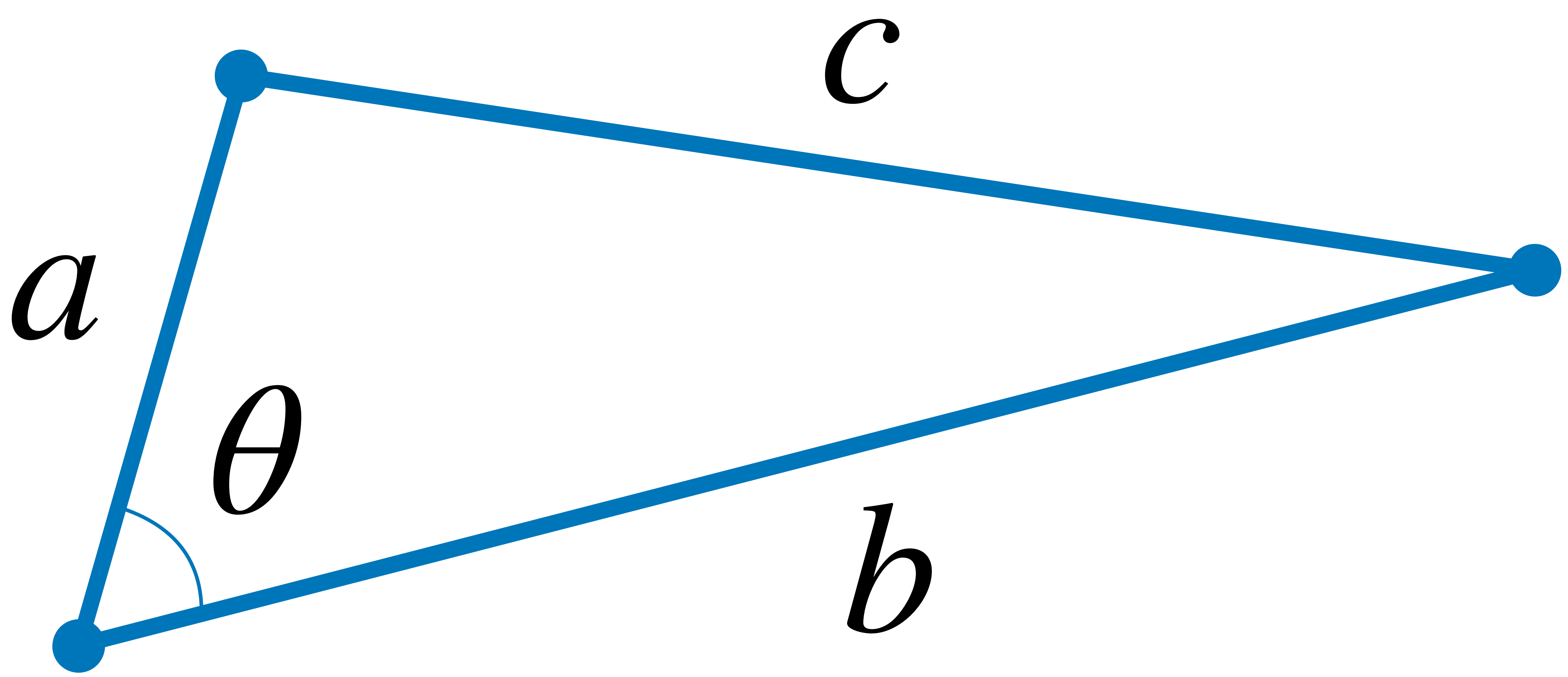


Theorem.

$$\boxed{c^2 = a^2 + b^2} - 2ab\cos\theta$$

Generalized the Pythagorean Theorem

Recall: Law of Cosines



Theorem.

0 exactly when $\theta = 90^\circ$

$$\boxed{c^2 = a^2 + b^2} \boxed{- 2ab\cos\theta}$$

Generalized the Pythagorean Theorem

Recall: Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

direction-only (scale/lenght-independent)

$$\cos\theta = \left\langle \frac{\mathbf{u}}{||\mathbf{u}||}, \frac{\mathbf{v}}{||\mathbf{v}||} \right\rangle = \frac{\langle \vec{u}, \vec{v} \rangle}{||\vec{u}||\,||\vec{v}||}$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations

Orthogonality (Perpendicularity)

A Simpler Fundamental Question

How do we determine if angle between any two vectors is 90°?

Recall: Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

$\cos 90^\circ = \cos \pi/2 = 0$

$$\cos\theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

Orthogonality

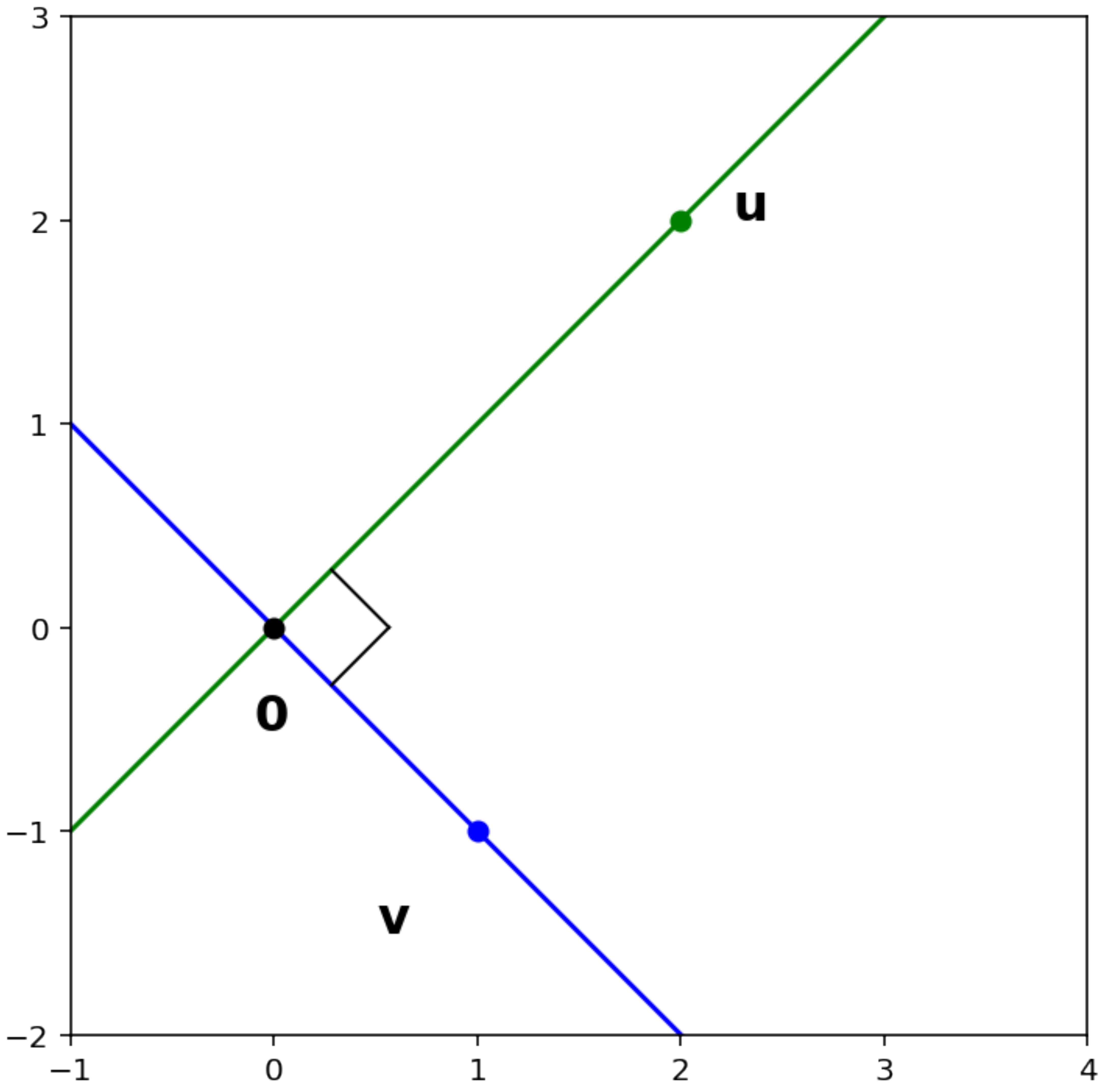
Definition. Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

This definition gives an easy computational way to determine orthogonality.

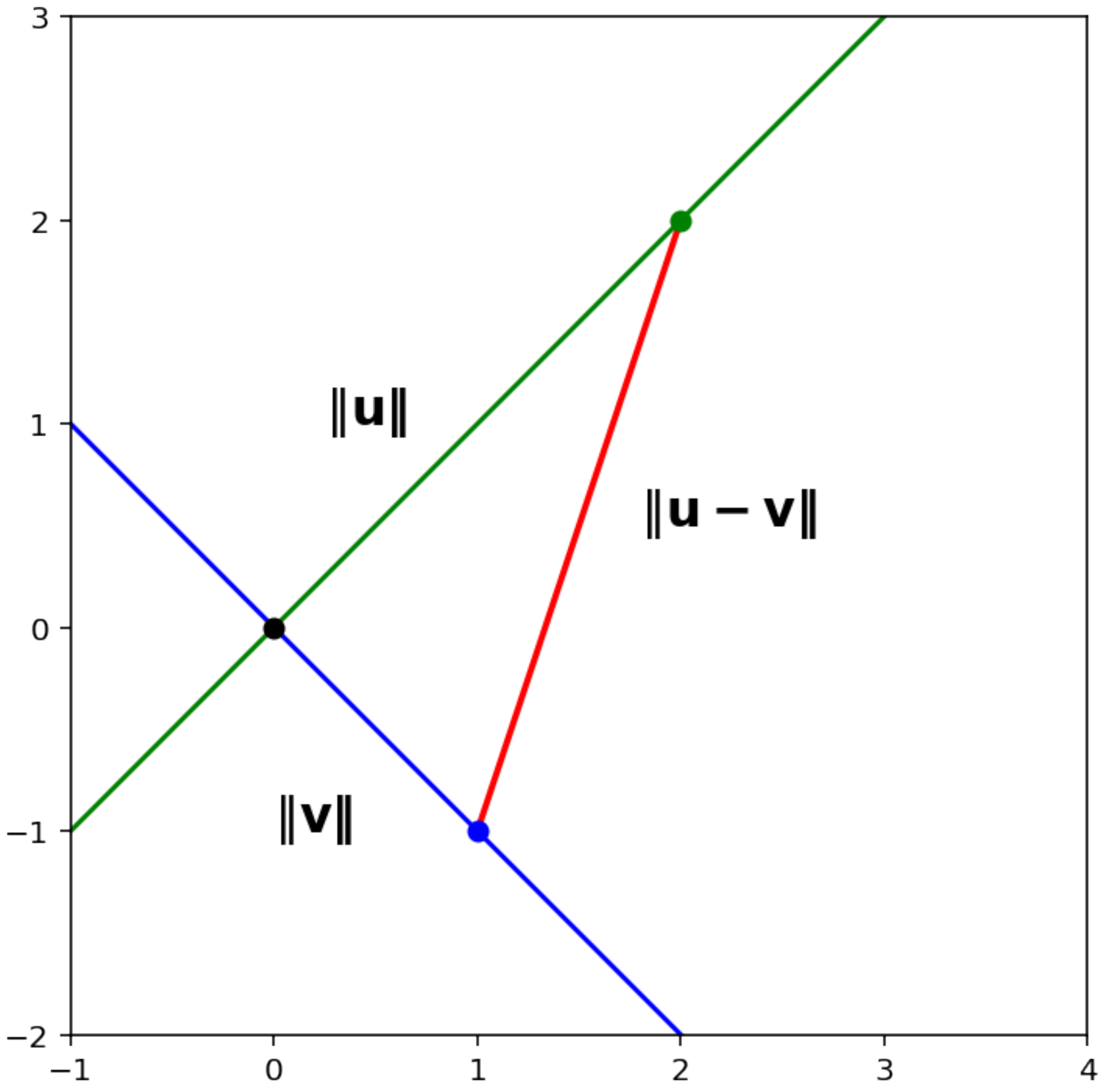
Example.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 - 2 = 0$$

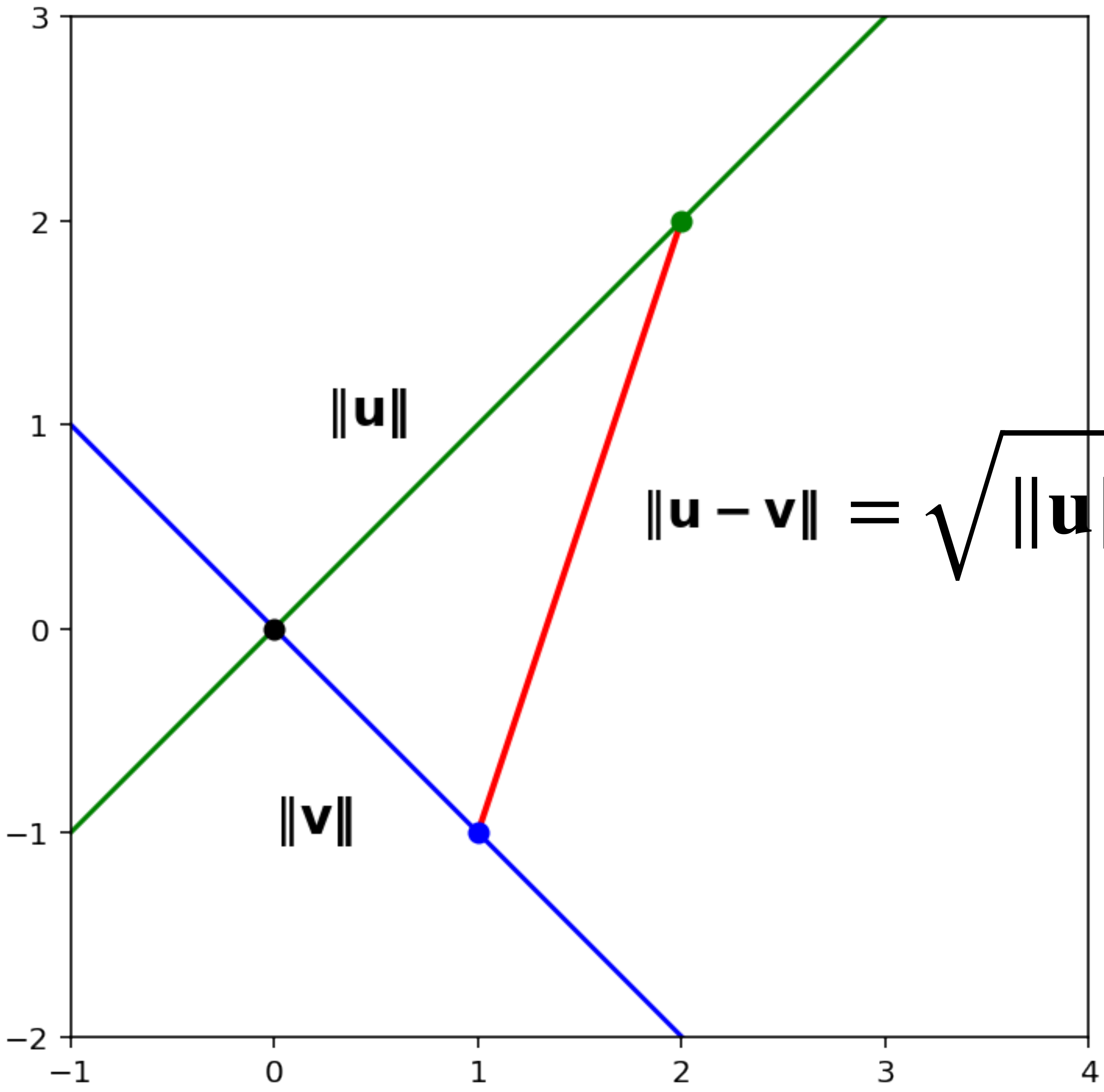
Derivation by Picture



Derivation by Picture



Derivation by Picture



$$\|\mathbf{u}-\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}$$

(Pythagoras)

Derivation by Algebra

u and **v** are orthogonal exactly when

$$\cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} = \|\mathbf{u} - \mathbf{v}\|^2$$

Let's simplify this a bit:

$$= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle$$

$$= \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\langle \vec{u}, \vec{v} \rangle$$

$$\Leftarrow$$

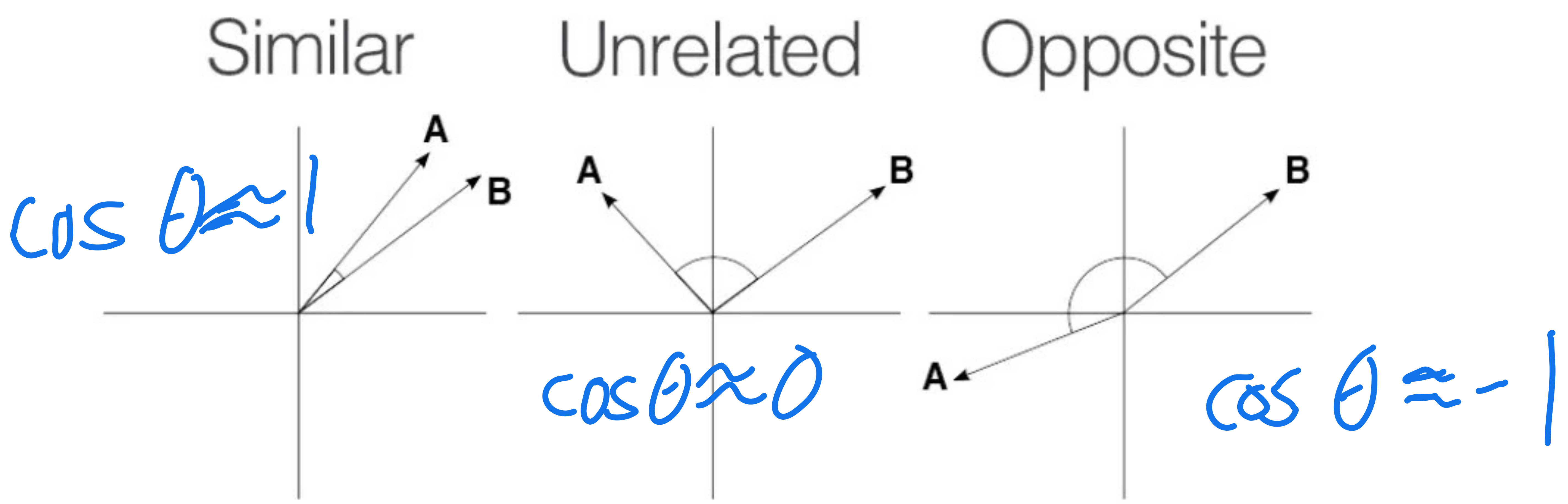
$$-2\langle \vec{u}, \vec{v} \rangle = 0$$

$$\Updownarrow$$

$$\langle \vec{u}, \vec{v} \rangle = 0$$

Application: Cosine Similarity

High Level



Data points are <u>very big vectors</u>.

Similar vectors "point in nearly the same direction."

https://medium.com/@milana.shxanukova15/cosine-distance-and-cosine-similarity-a5da0e4d9ded

Example: Netflix Users

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix} \quad \text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix} \quad \text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{l} \texttt{comedy} \\ \texttt{drama} \\ \texttt{horror} \\ \texttt{romance} \end{array}$$

A Netflix user might be represented as a vectors whose ith entry is the number of movies they've watched in a particular genre.

Who are more likely to share similar interests in movies?

Cosine Similarity

$-1 \leq \cos\theta \leq 1$

Definition. The **cosine similarity** of two vectors is the cosine of the angle between them.

If its close to 0, then two Netflix users watch very different movies.

If its close to 1, then two Netflix users watch very similar movies.

Example: Netflix Users

```
comedy
drama
horror
romance
```

$$\text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix}$$

$$\text{sim}(\text{user}_2, \text{user}_3) \approx 0.52$$

$$\text{sim}(\text{user}_1, \text{user}_2) \approx 0.92$$

$$\text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix}$$

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{sim}(\text{user}_1, \text{user}_3) \approx 0.32$$

Other Examples

- ***Document similarity***
  - Documents $\mapsto$ word count vectors
  - Similar documents should use similar words
- ***Word2Vec***
  - Words $\mapsto$ vector *somehow*
  - This underlies modern natural language processing (NLP)

Recall: Orthogonality

Definition. Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Orthogonal and perpendicular are the same thing.

With inner product we can...

- Given a vector we can determine its <u>length</u>
- Given two points (vectors) we can determine the <u>distance</u> between them
- Given two vectors we can determine the <u>angle</u> between them

Orthogonal Sets

Orthogonal Sets

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ of vectors from R^n is an **orthogonal set** if every pair of distinct vectors is orthogonal: if $i \neq j$ then

$$\langle u_i, u_j \rangle = 0$$

Each vector is pairwise/mutually perpendicular

Example

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \cdot \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} = 0$$

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

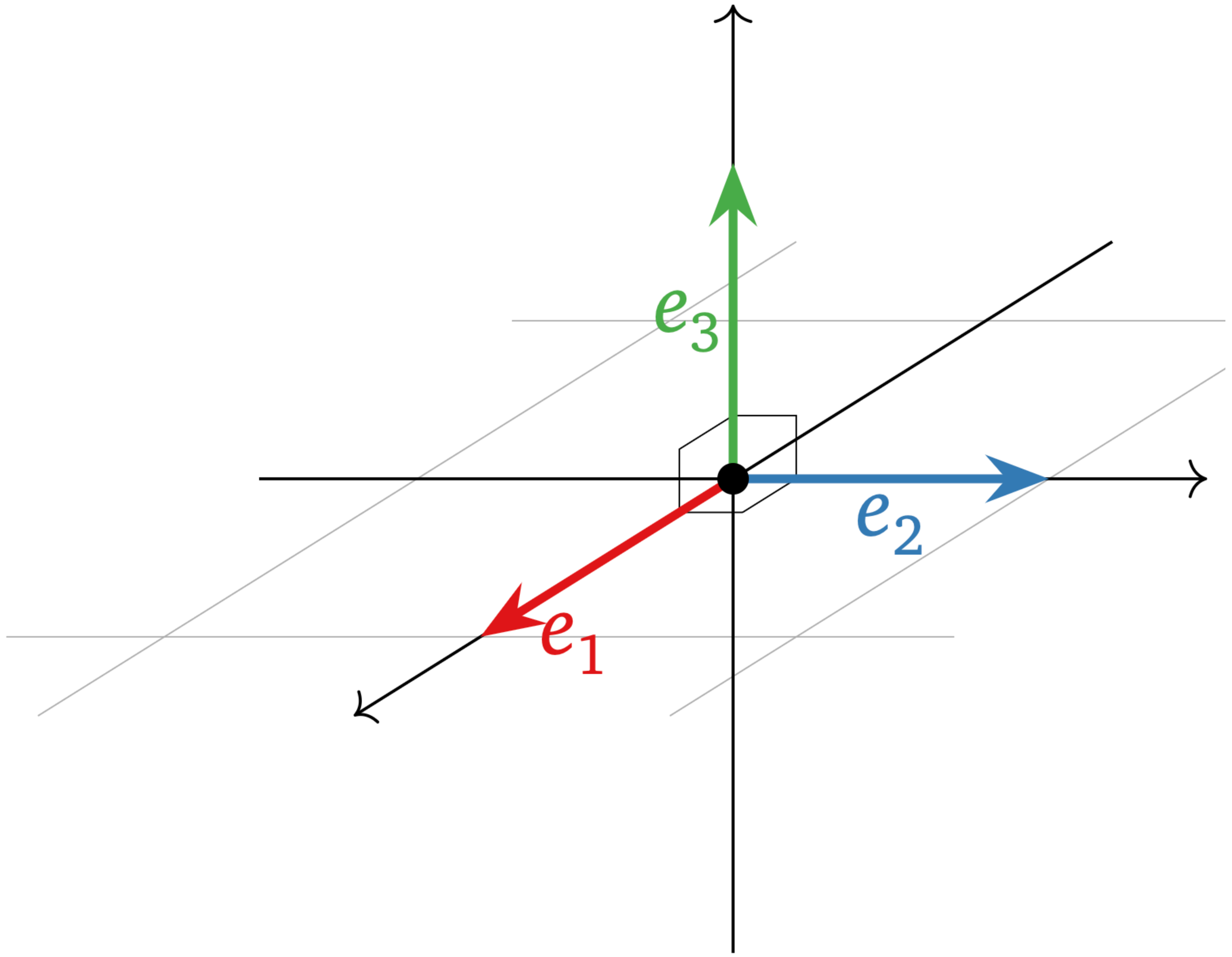
Verify: $\vec{u}_1 \cdot \vec{u}_2 = -3+2+1=0$

$\vec{u}_1 \cdot \vec{u}_3 = -3/2 - 2 + 7/2 = 0$

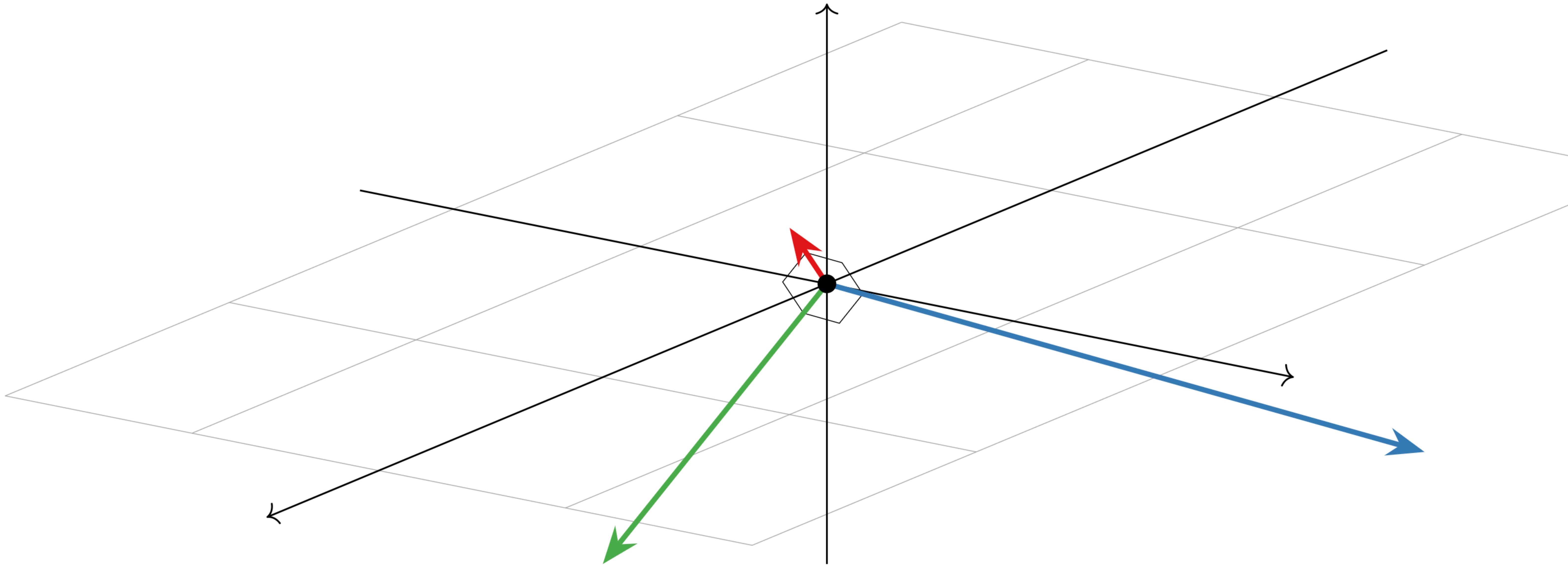
$\vec{u}_2 \cdot \vec{u}_3 = 1/2 - 4 + 7/2 = 1/2 - 8/2 + 7/2 = 0$

What do orthogonal sets look like?

The Picture



the standard basis forms a
"centered" orthogonal set



an orthogonal set is like
the standard basis *after*
some rotations and scalings
reflections

https://textbooks.math.gatech.edu/ila/orthogonal-sets.html

Orthogonal Sets and Independence

Theorem. If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set of *nonzero* vectors from R^n, then it is linearly independent

Verify: $\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_k \vec{u}_k = \vec{0}$ (dot prod. with $\vec{u}_1$)

$\alpha_1(\vec{u}_1 \cdot \vec{u}_1) + \alpha_2(\cancel{\vec{u}_2 \cdot \vec{u}_1}) + \cdots + \alpha_k(\cancel{\vec{u}_k \cdot \vec{u}_1}) = \vec{0} \cdot \vec{u}_1$

$\alpha_1 \|\vec{u}_1\|^2 = 0 \Rightarrow \alpha_1 = 0$

(same arg with general $\vec{u}_j \Rightarrow \alpha_j = 0$) only trivial sol'n

The Takeaway

If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set, then it is a **basis** for $span\{u_1, u_2, \ldots, u_k\}$

Orthogonal Basis

Definition. An **orthogonal basis** for a subspace W of R^n is a basis for W which is also an orthogonal set.

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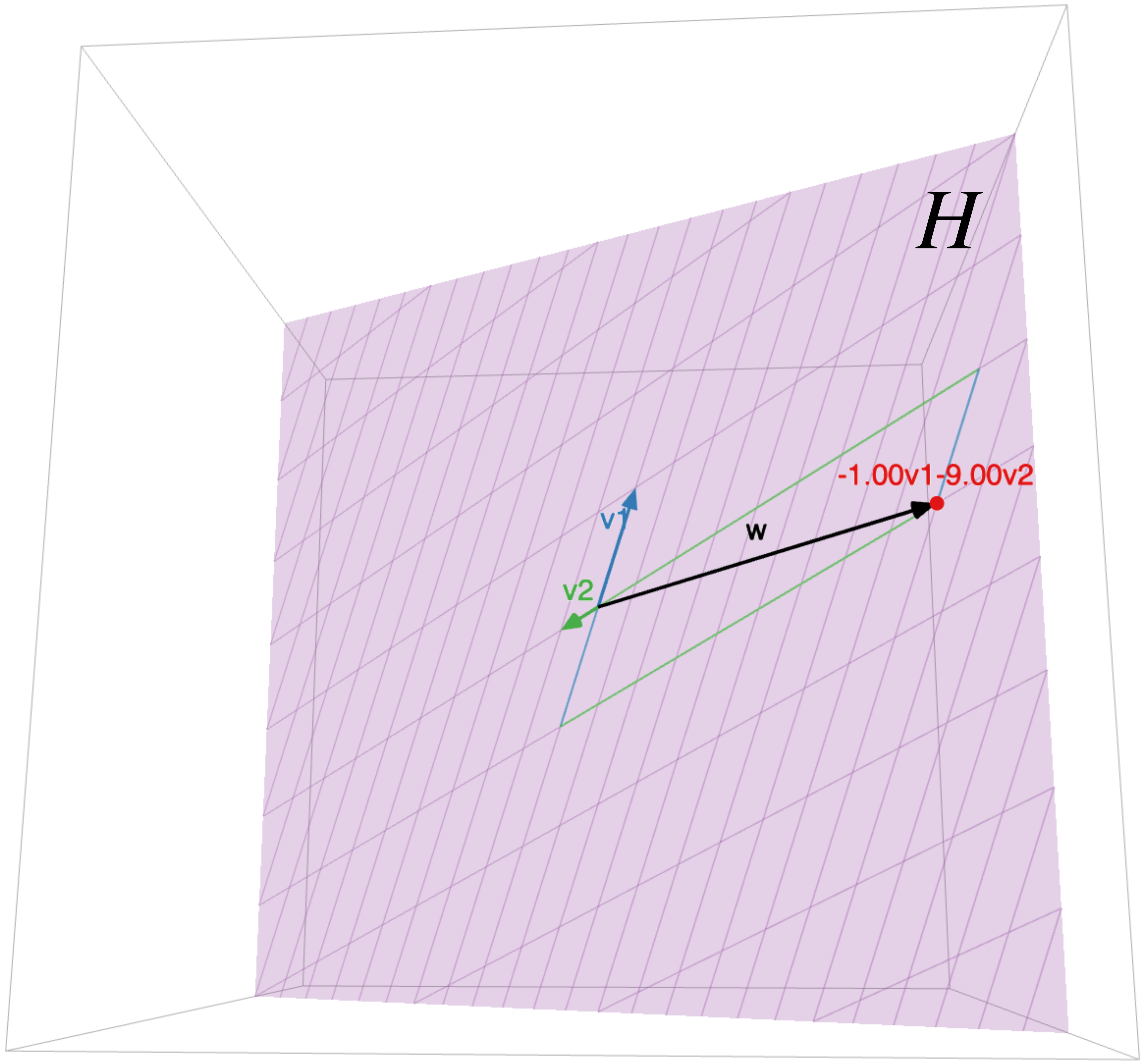

v_1 and v_2 form a basis of H

https://textbooks.math.gatech.edu/ila/spans.html

Orthogonal Basis

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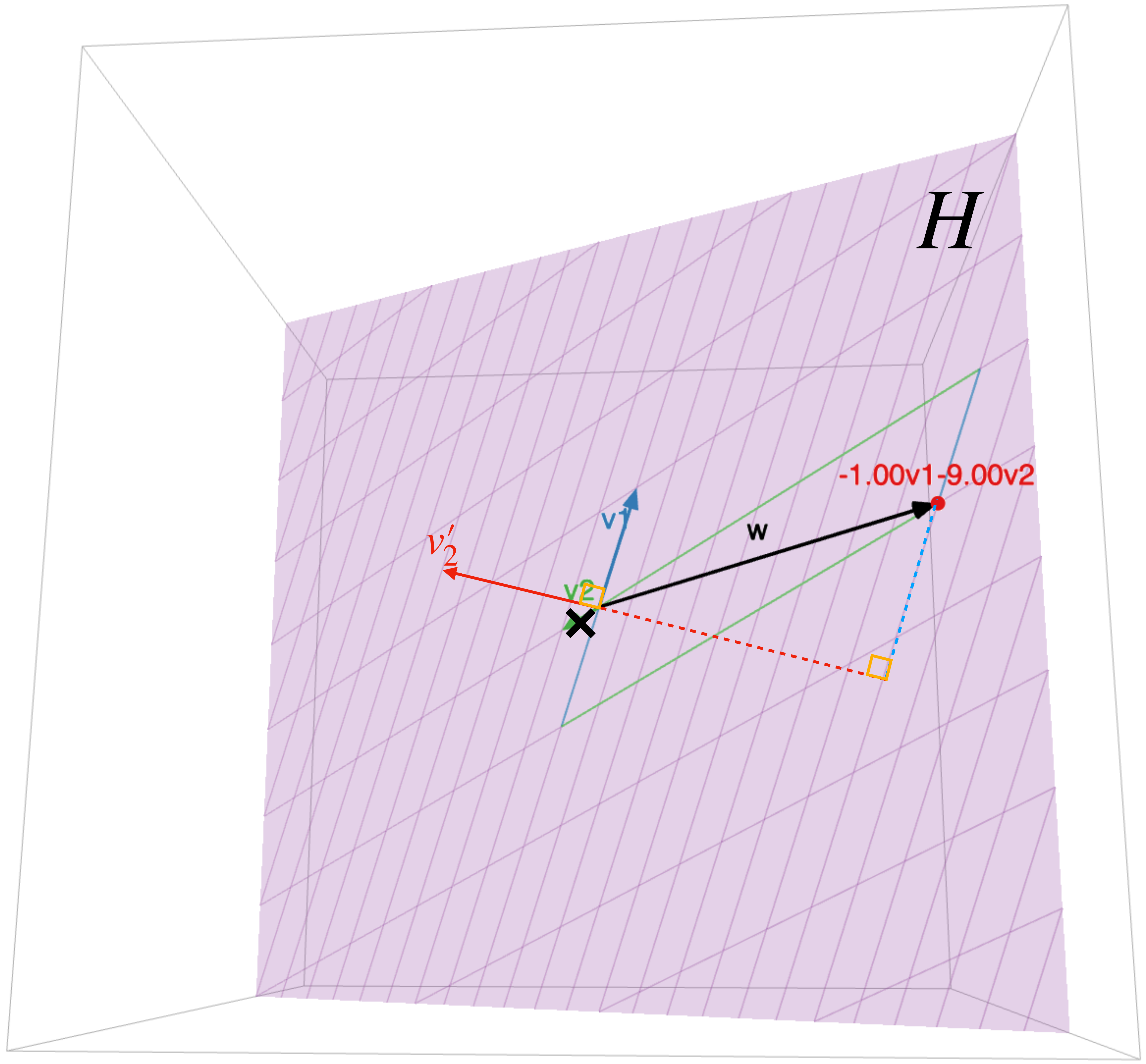

v_1 and v_2 form a basis of H

v_1 and v_2' form an **orthogonal** basis of H

https://textbooks.math.gatech.edu/ila/spans.html

What's nice about an orthogonal basis?

Recall: How To: Bases

Recall: How To: Bases

Question. Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

unique

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Recall: How To: Bases

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$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. Solve the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_p\mathbf{u}_p = \mathbf{w}$$

$$\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \vec{w} \end{bmatrix}$$

by Gaussian elimination, matrix inversion, etc.

Recall: How To: Bases

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Solution. Solve the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_p\mathbf{u}_p = \mathbf{w}$$

by Gaussian elimination, matrix inversion, etc.

This takes work

Orthogonal Bases and Linear Combinations

Theorem. For an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, if $\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ then for $j = 1, \ldots, p$

just use dot products

$$c_j = \frac{\mathbf{y}^T\mathbf{u}_j}{\mathbf{u}_j^T\mathbf{u}_j}$$

Verify: $\vec{y} \cdot \vec{u}_j = c_1(\vec{u}_1 \cdot \vec{u}_j) + c_2(\vec{u}_2 \cdot \vec{u}_j) + \cdots + c_p(\vec{u}_p \cdot \vec{u}_j)$

$= c_j \|\vec{u}_j\|^2 \Rightarrow c_j = \frac{\vec{y} \cdot \vec{u}_j}{\|\vec{u}_j\|^2}$

How To: Orthogonal Bases

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Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

How To: Orthogonal Bases

Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \dots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots c_p\mathbf{u}_p$$

Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

How To: Orthogonal Bases

Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

Much easier to compute.

Question

$$\vec{y}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -1/2 & 6 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 7/2 & -8 \end{array}\right]$$ slow

Express $[6 \;\; 1 \;\; (-8)]^T$ *as a linear combination of vectors in* $\{u_1, u_2, u_3\}$ *where* orthogonal set

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$$c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\|\vec{u}_1\|^2} = \frac{18+1-8}{9+1+1} = 1$$

$$c_2 = \frac{\vec{y}\cdot\vec{u}_2}{\|\vec{u}_2\|} = \frac{-6+2-8}{1+4+1} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\vec{y}\cdot\vec{u}_3}{\|\vec{u}_3\|^2} = \frac{-6/2 - 2 - 56/2}{1/4 + 4 + 49/4}$$

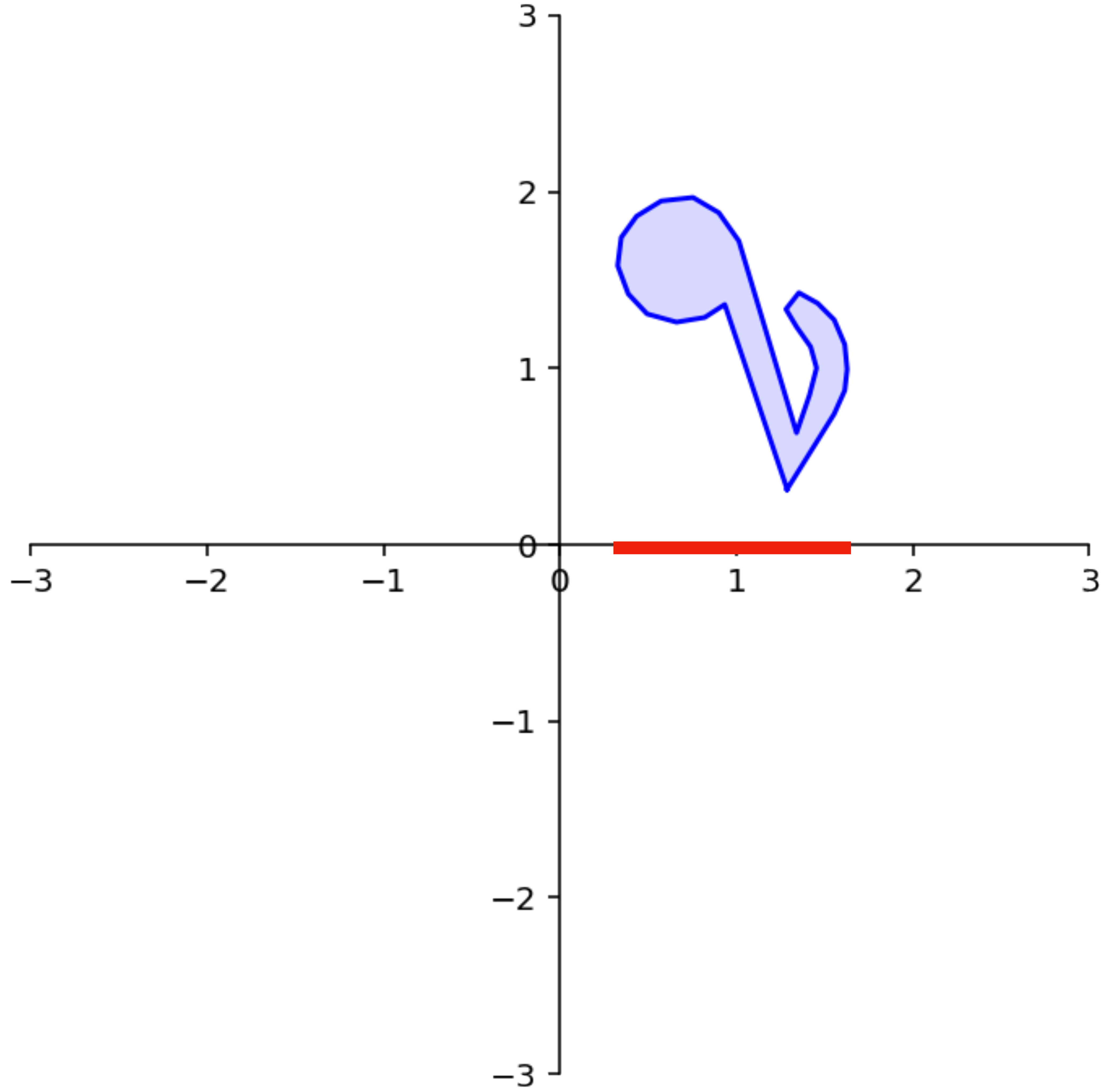
$$= \frac{-66/2}{66/4} = -\frac{4}{2} = -2$$

Answer: $u_1 - 2u_2 - 2u_3$

Orthogonal Projection

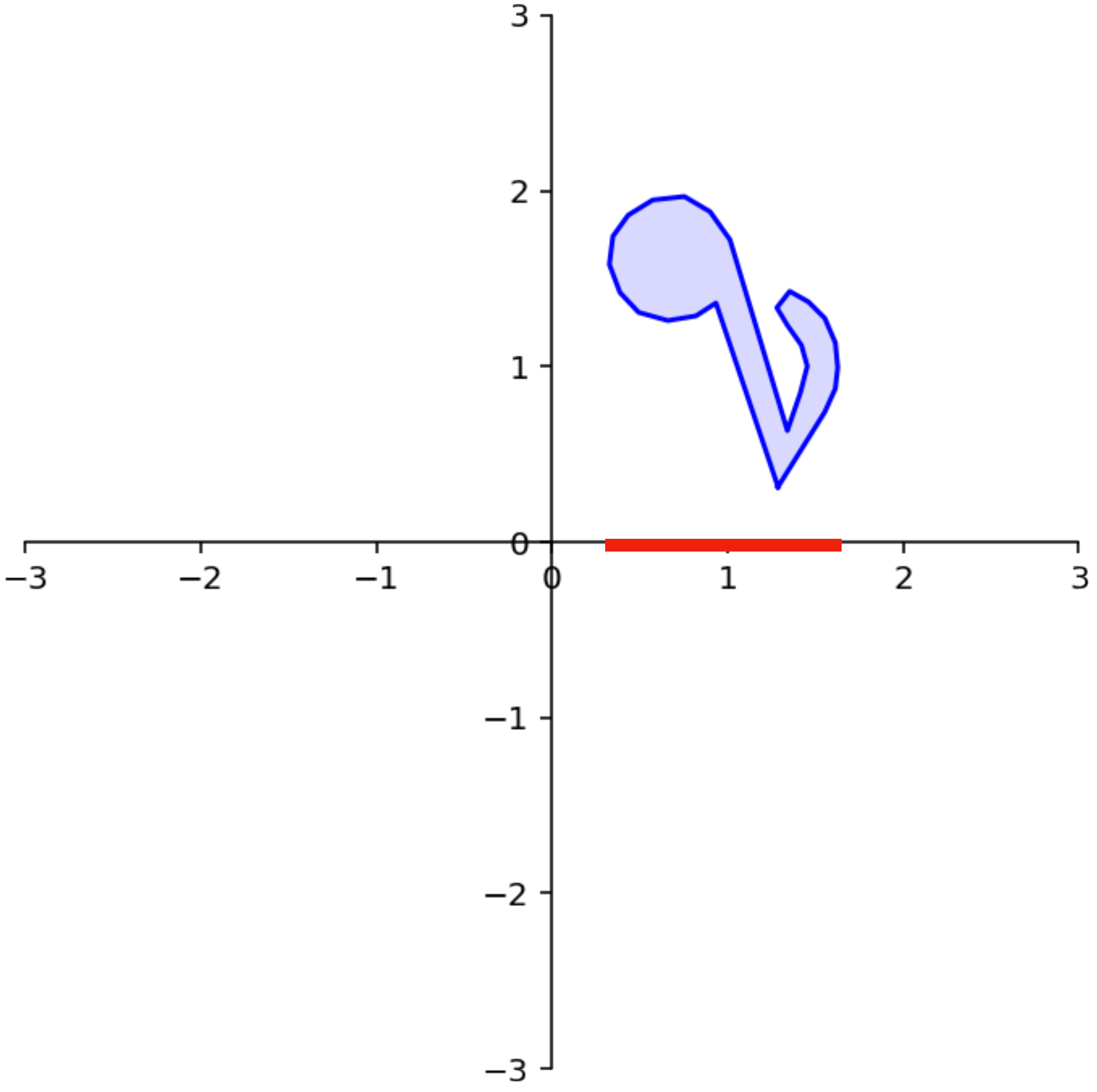
Why does that formula in the last example work?

Recall: Projection onto the x-axis



Recall: Projection onto the x-axis

We've seen simple projections in R^2



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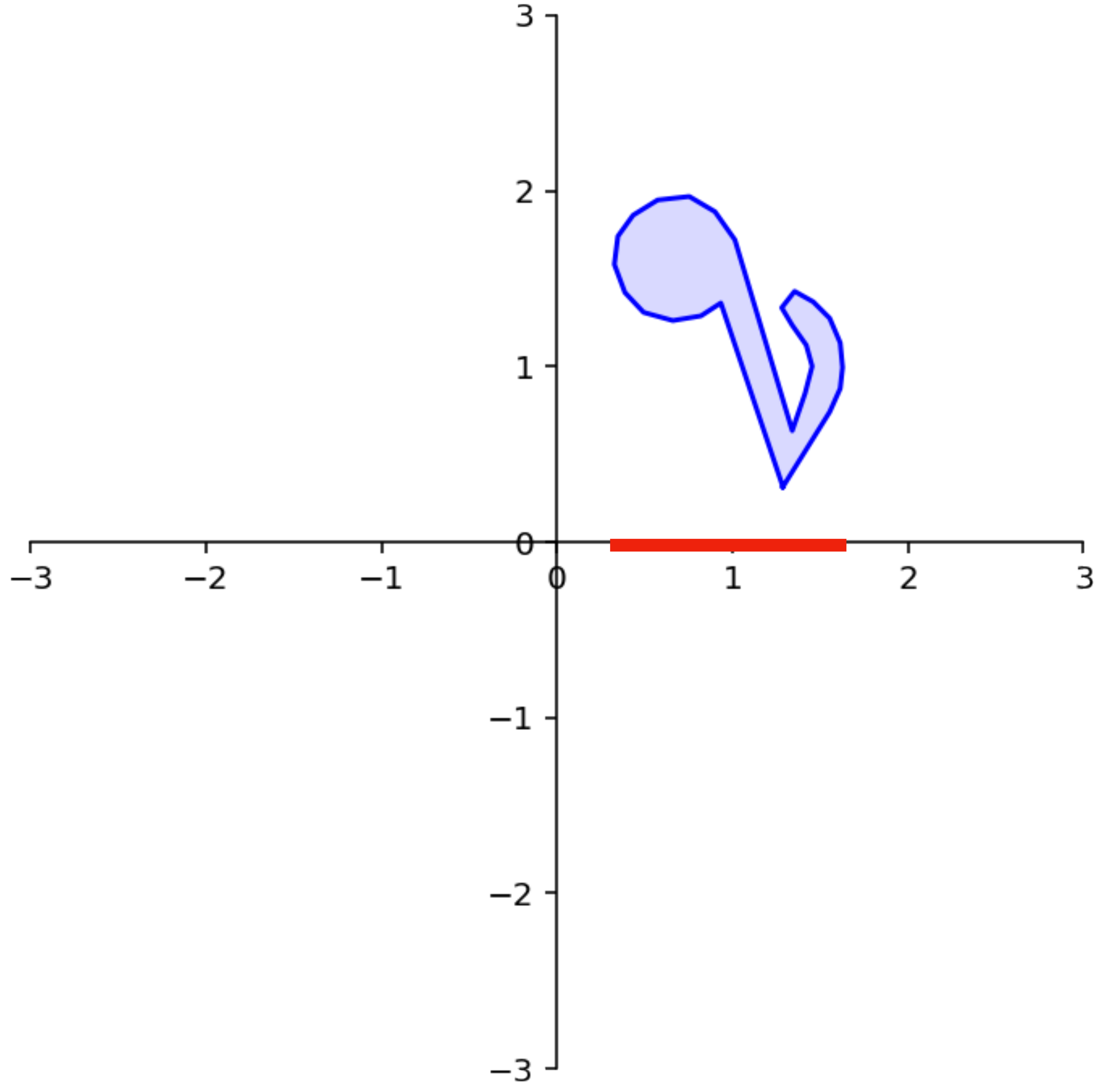
We're going to generalize this idea



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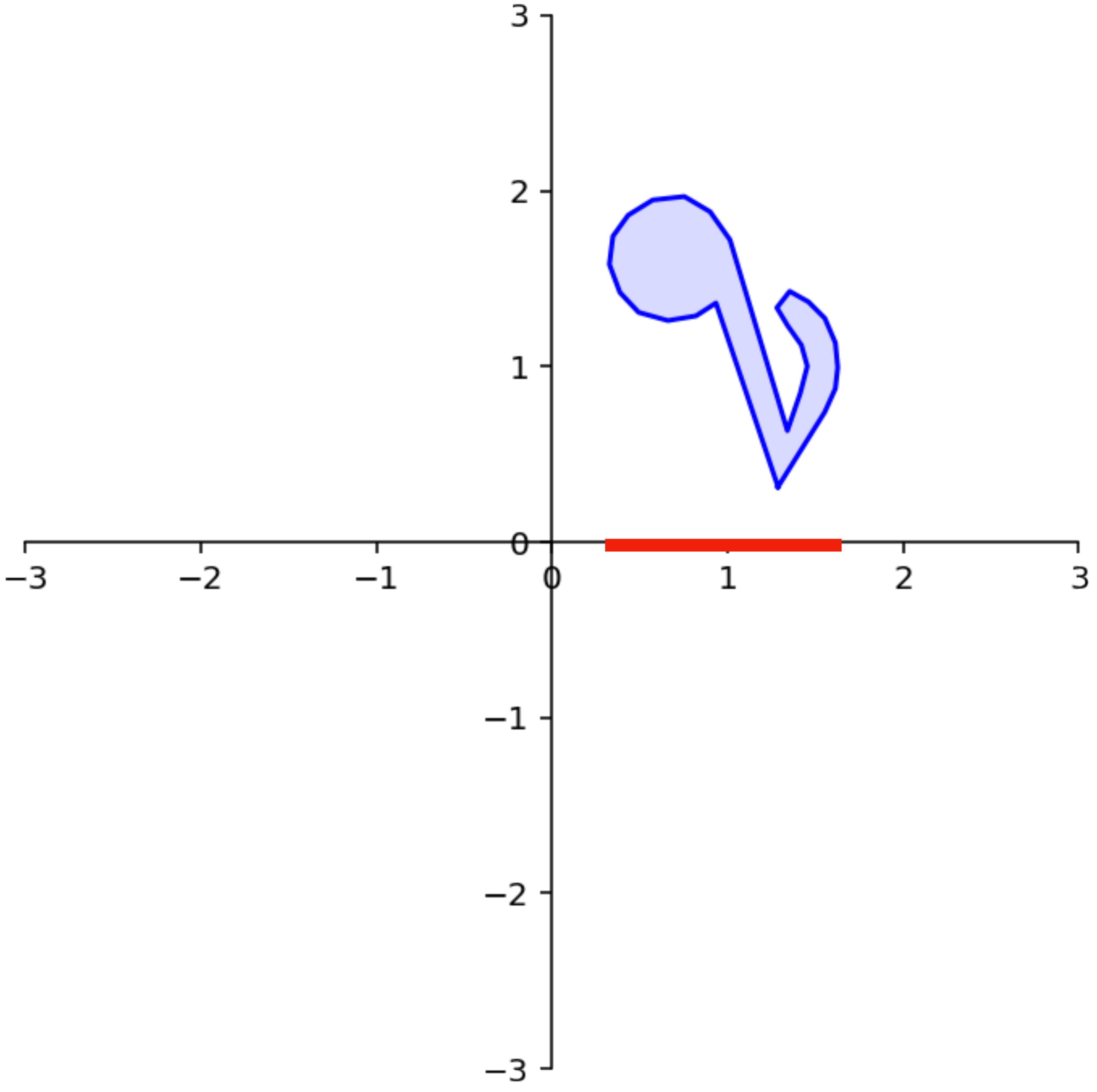
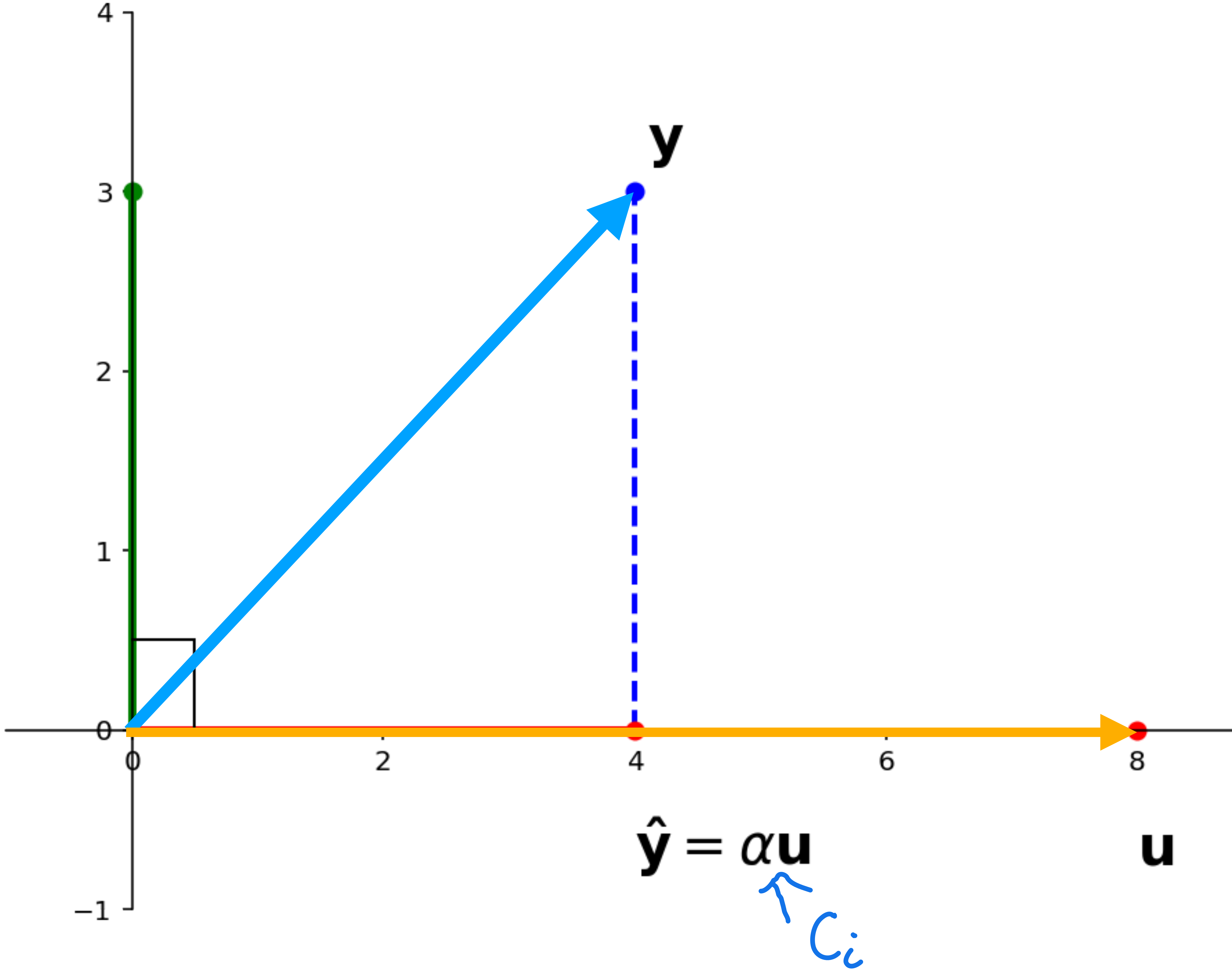
We're going to generalize this idea

What we really did was a kind of projection onto the basis vectors



Orthogonal Projection



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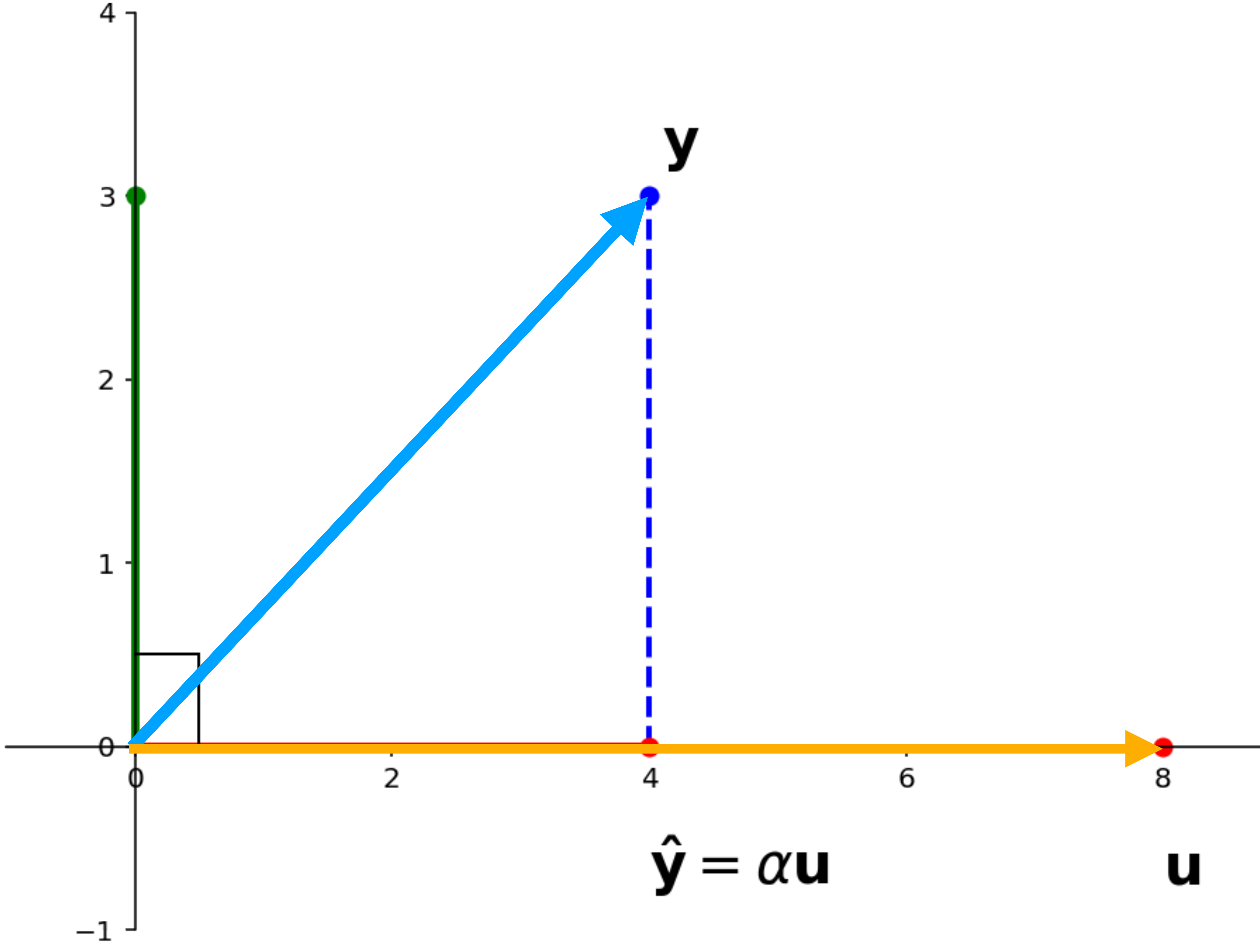
Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that



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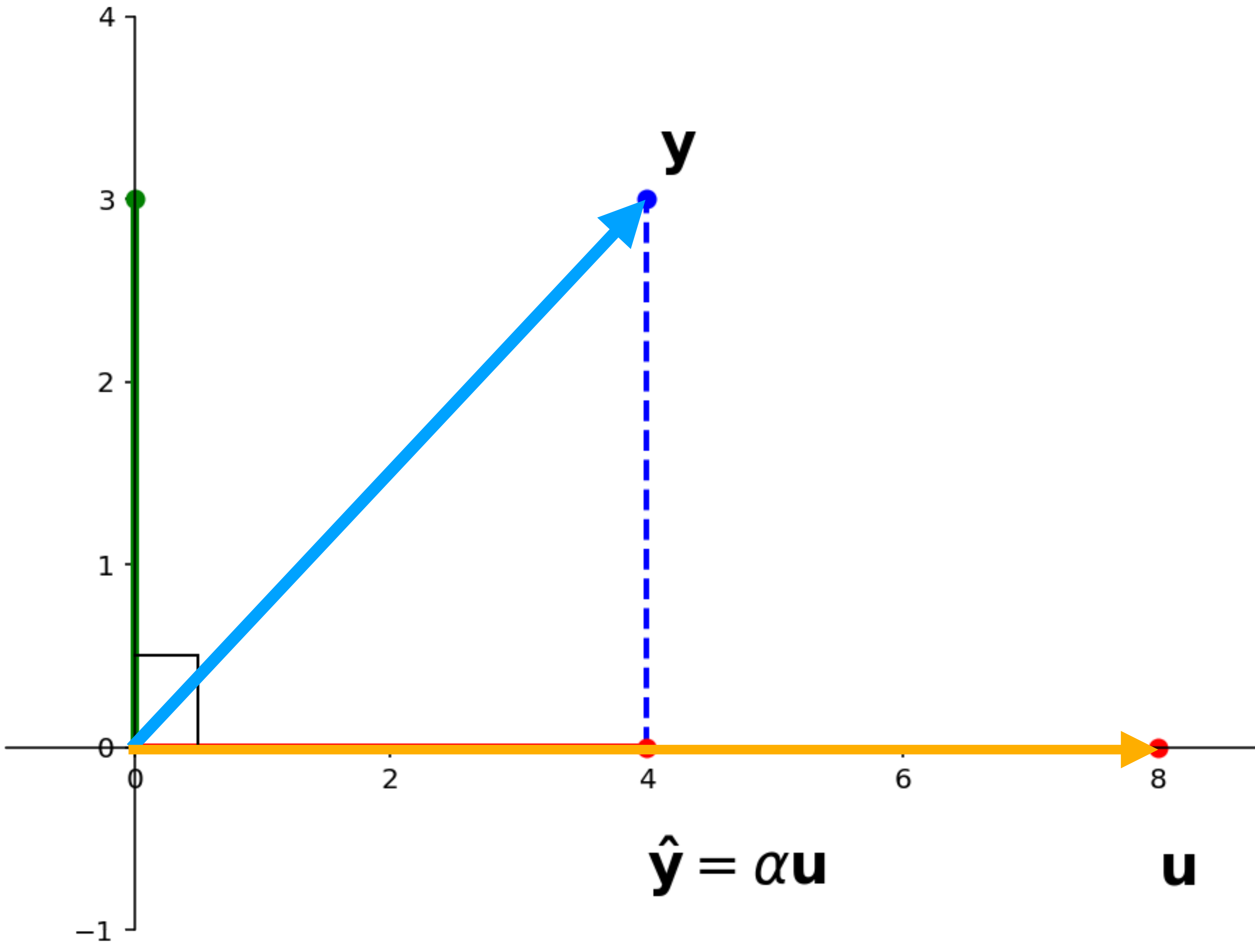

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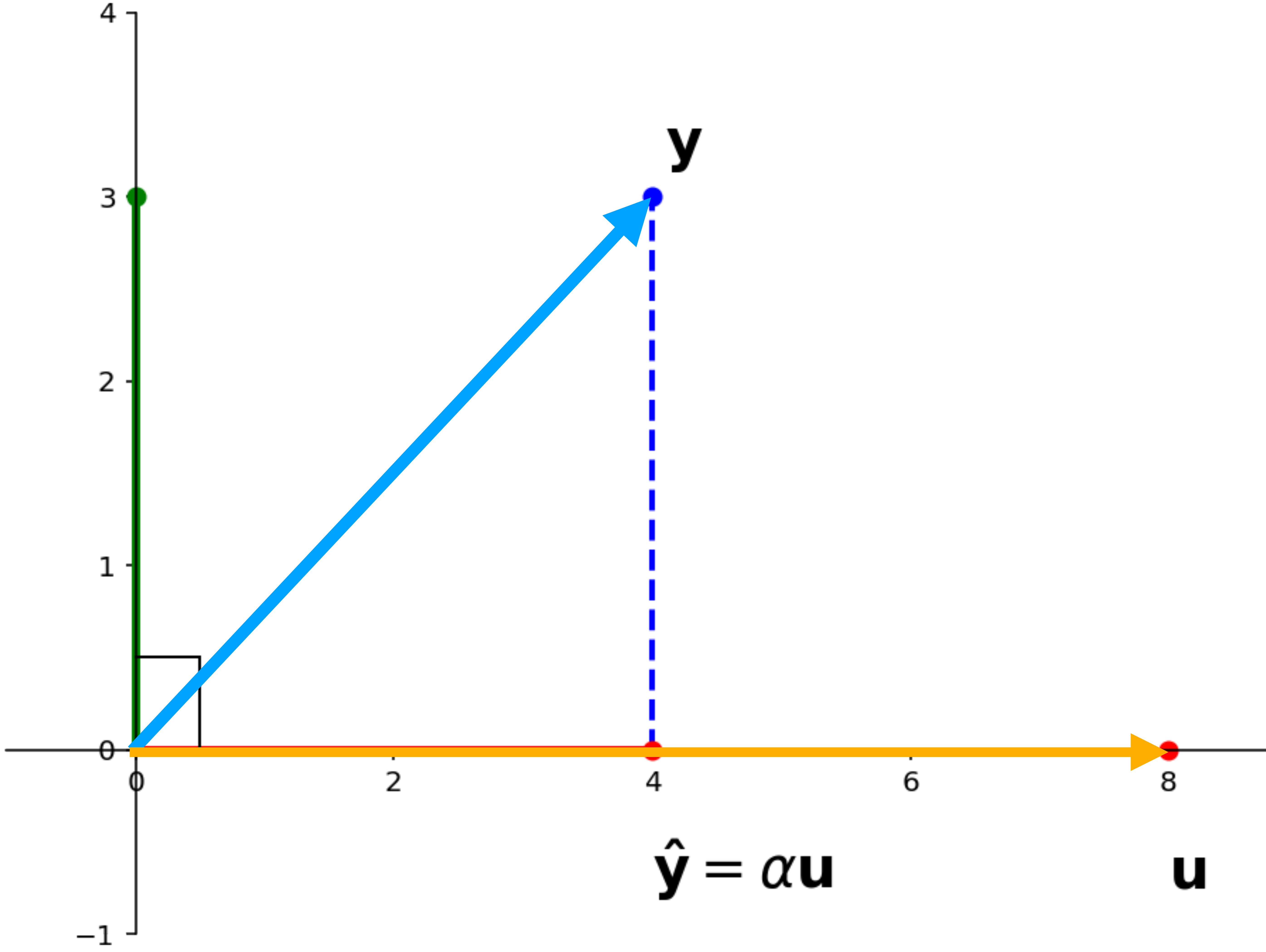

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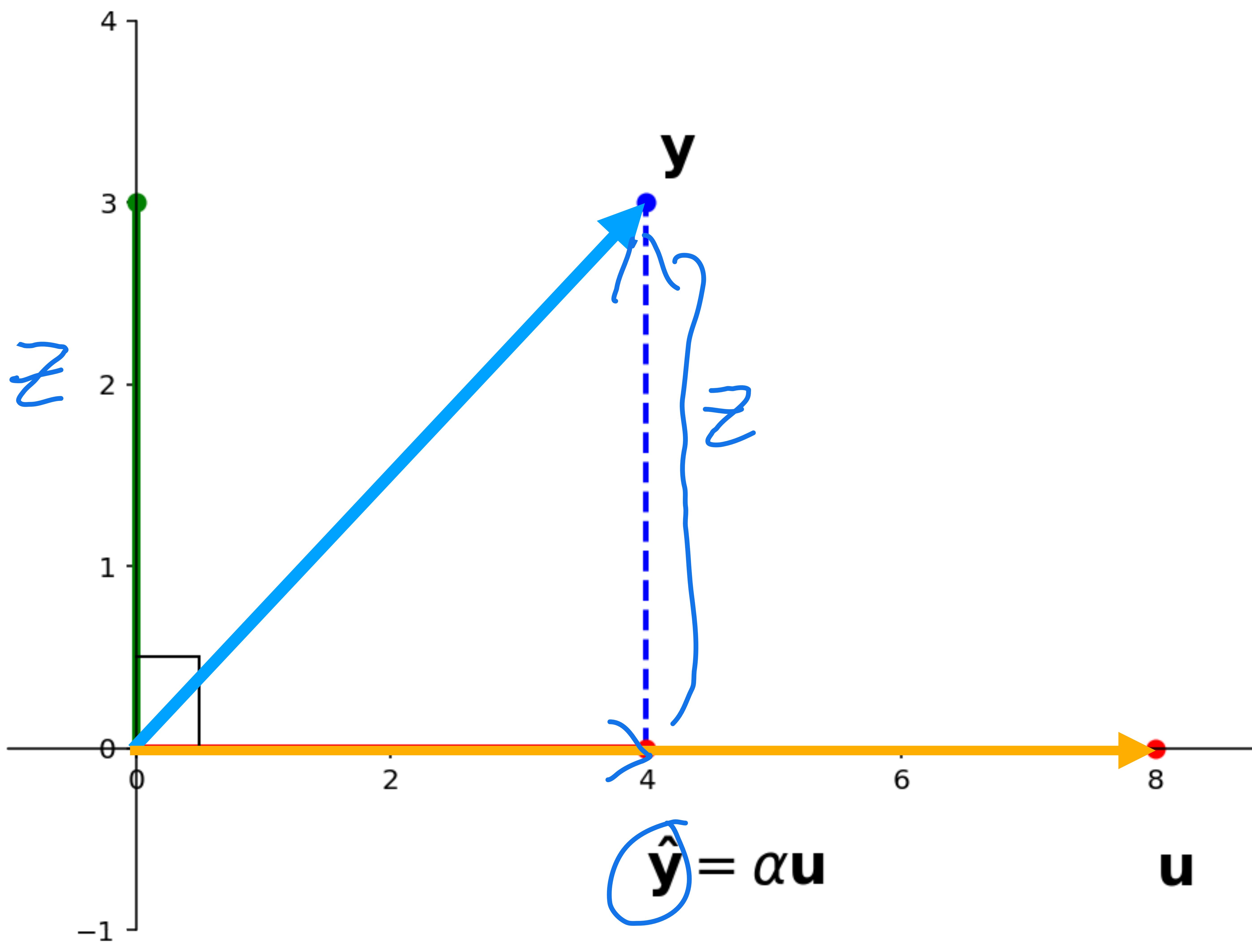

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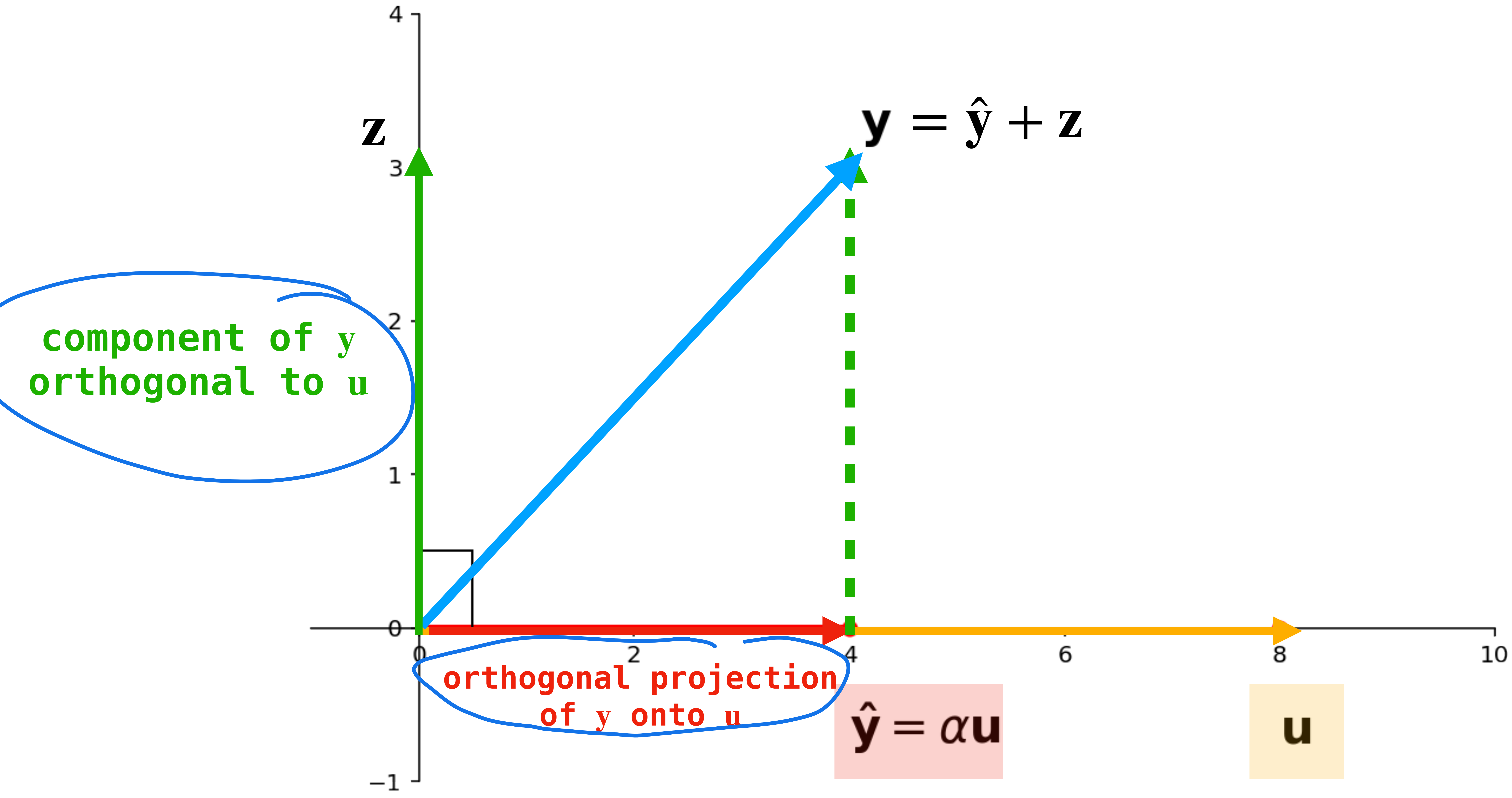

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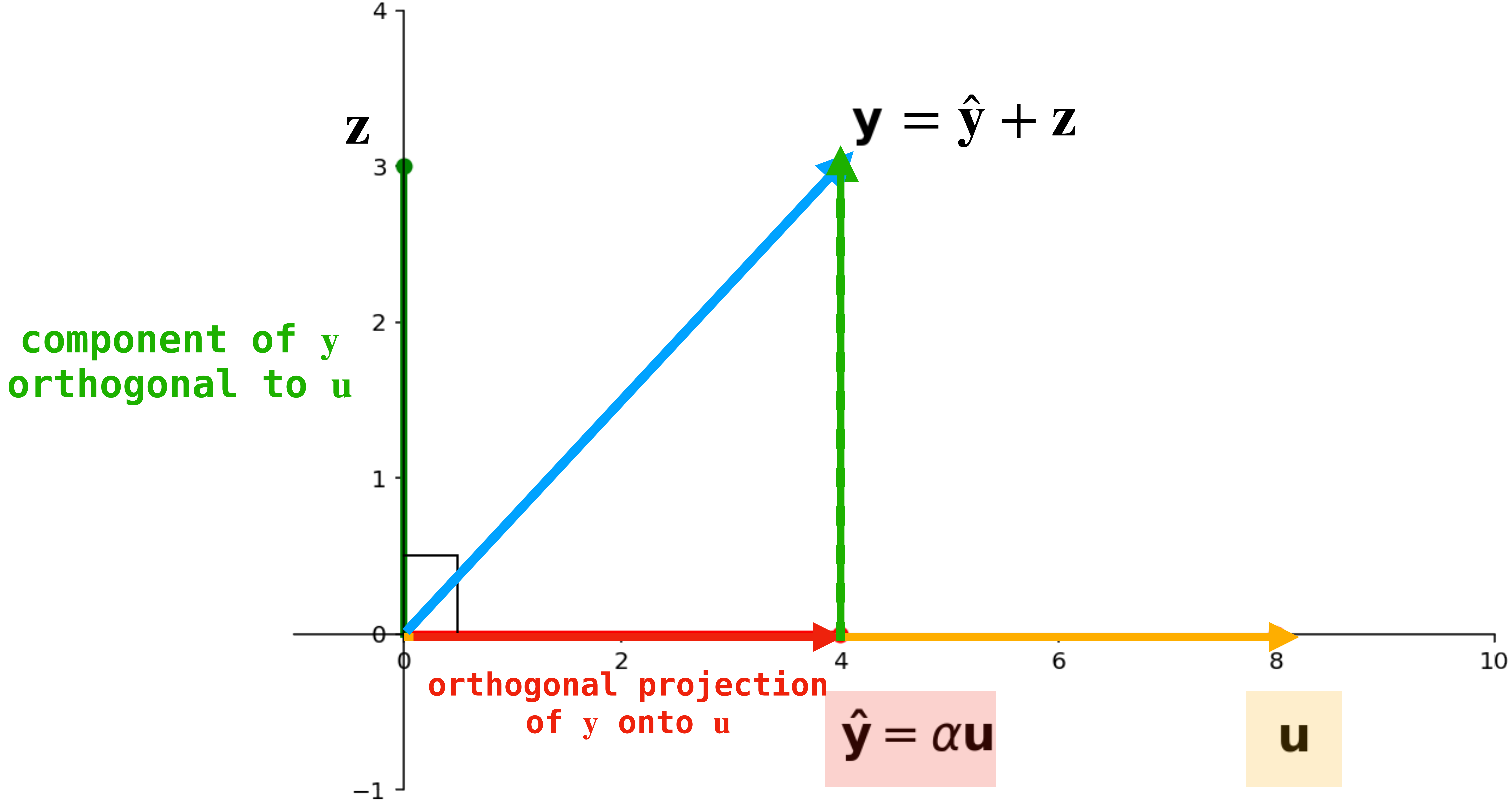
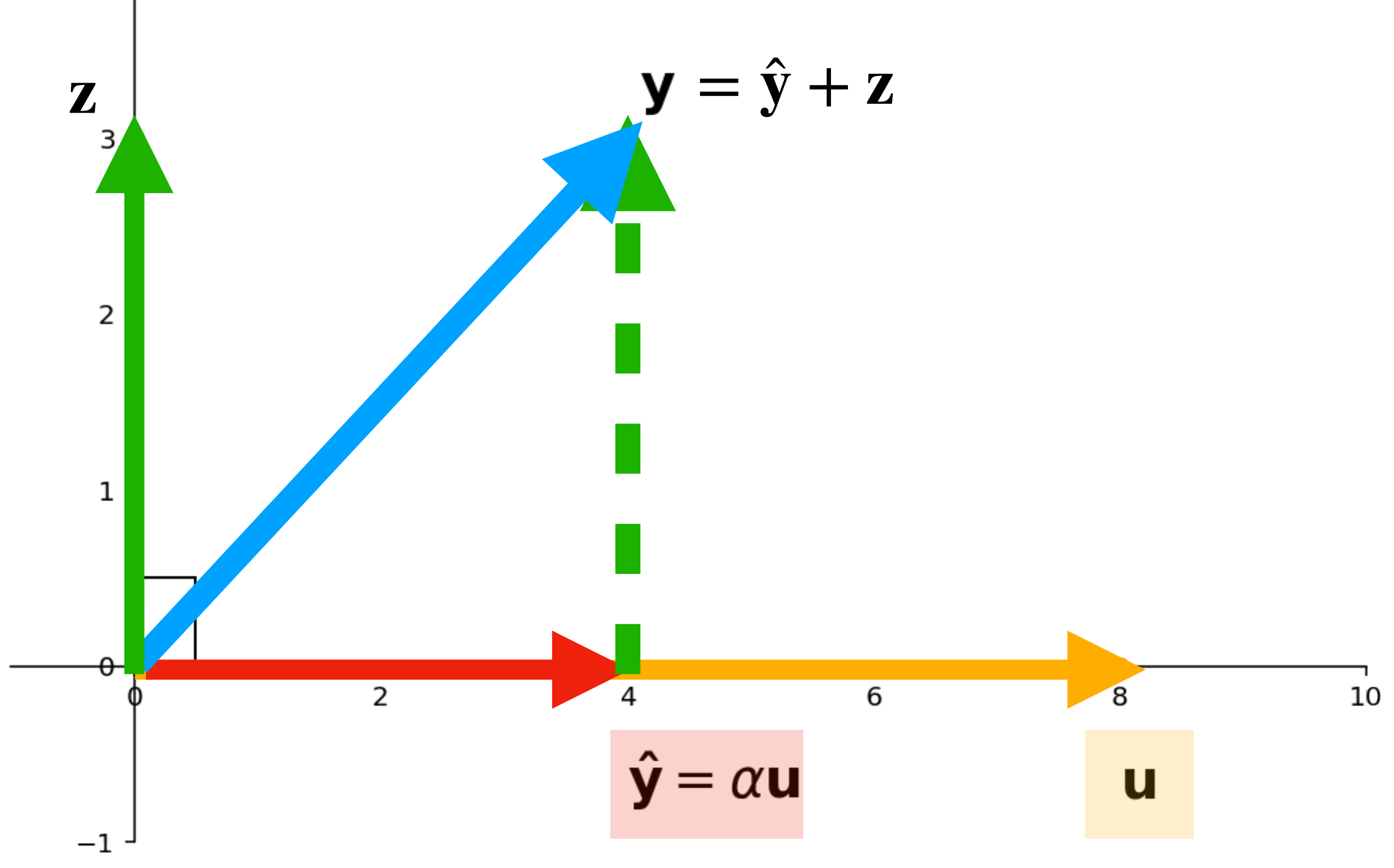


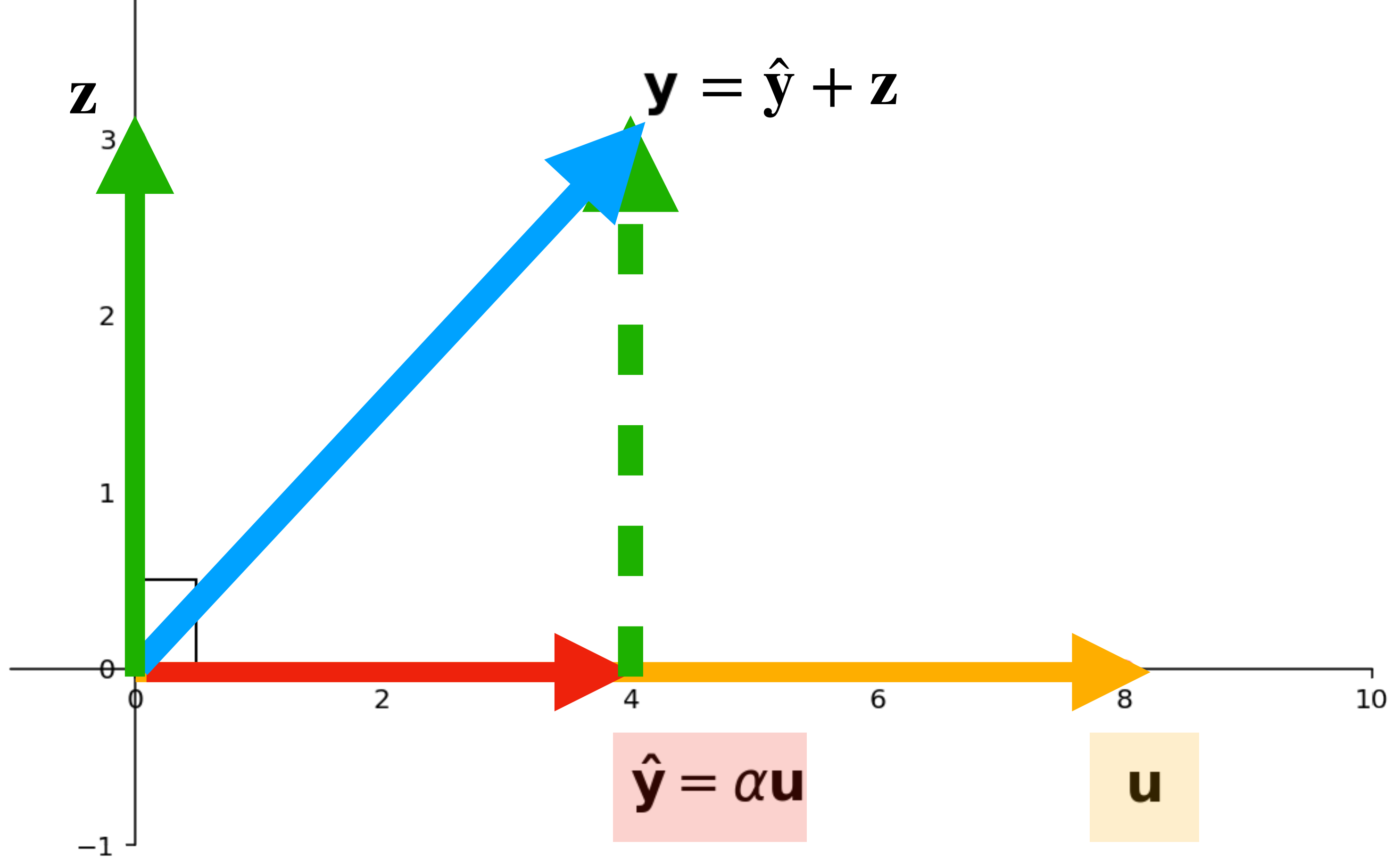
How do we find the orthogonal projection and orthogonal component?

What we know



$\mathbf{z}$
$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$
3
2
1
0
−1
0
2
4
6
8
10
$\hat{\mathbf{y}} = \alpha\mathbf{u}$
$\mathbf{u}$

What we know



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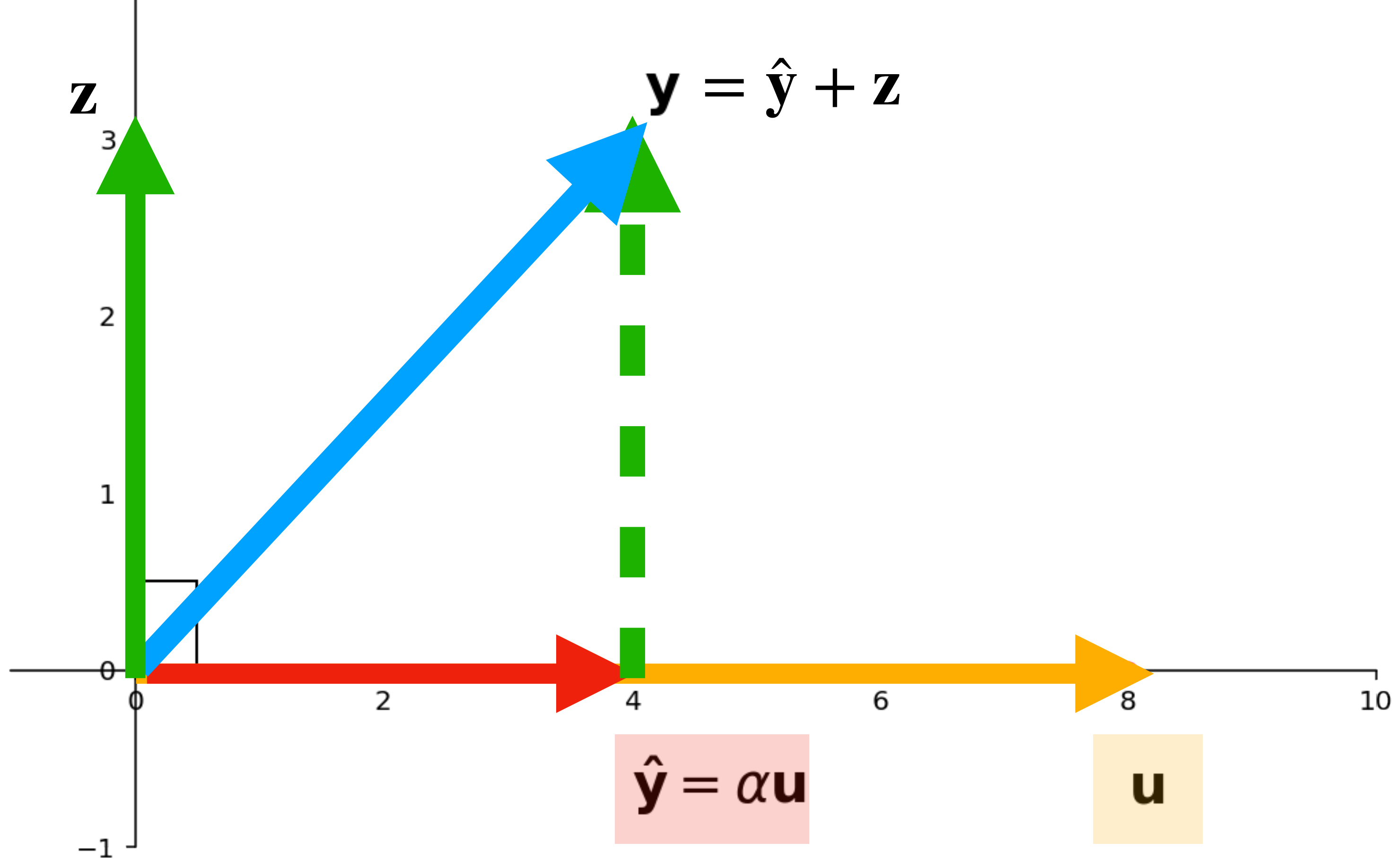
What we know



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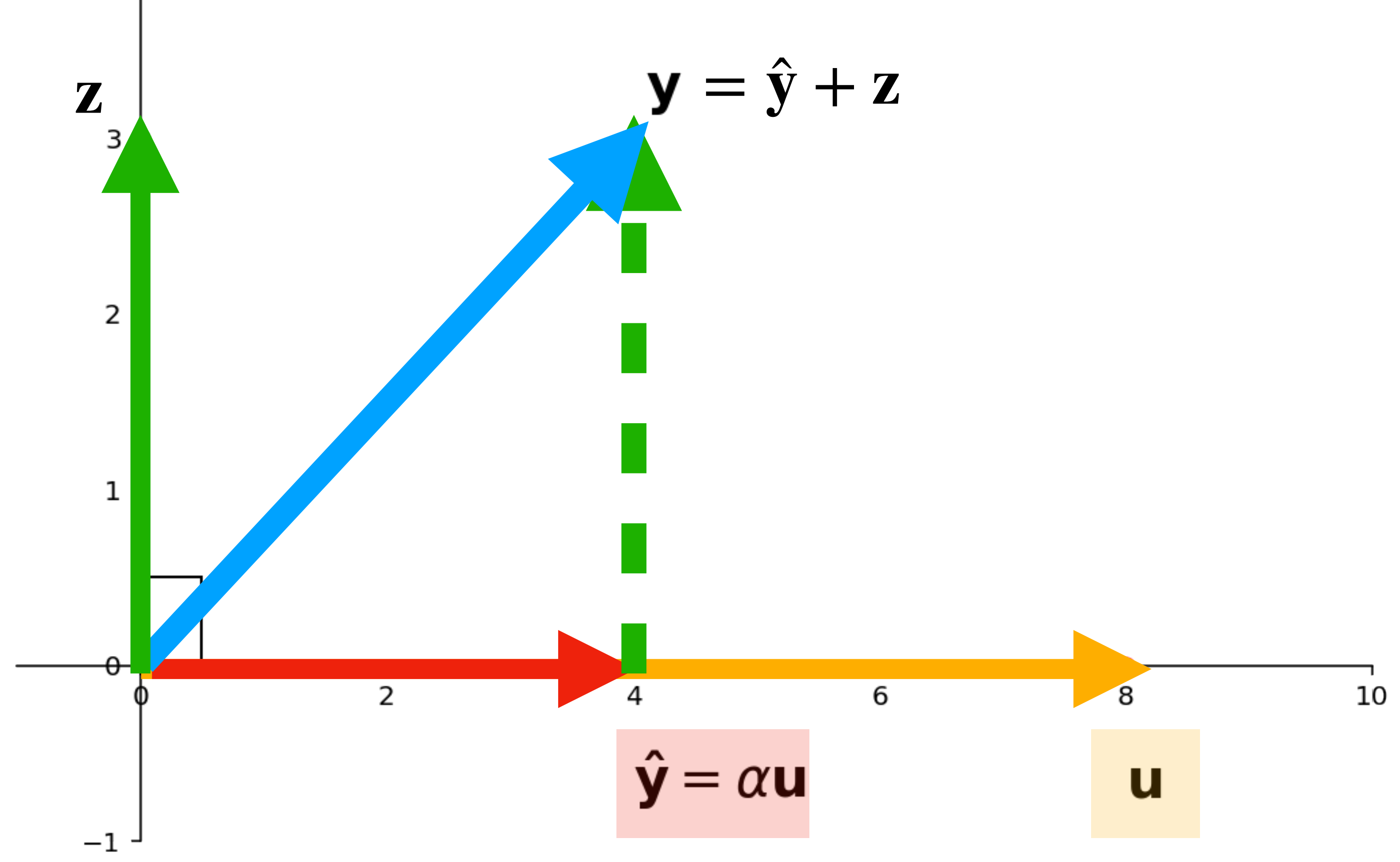

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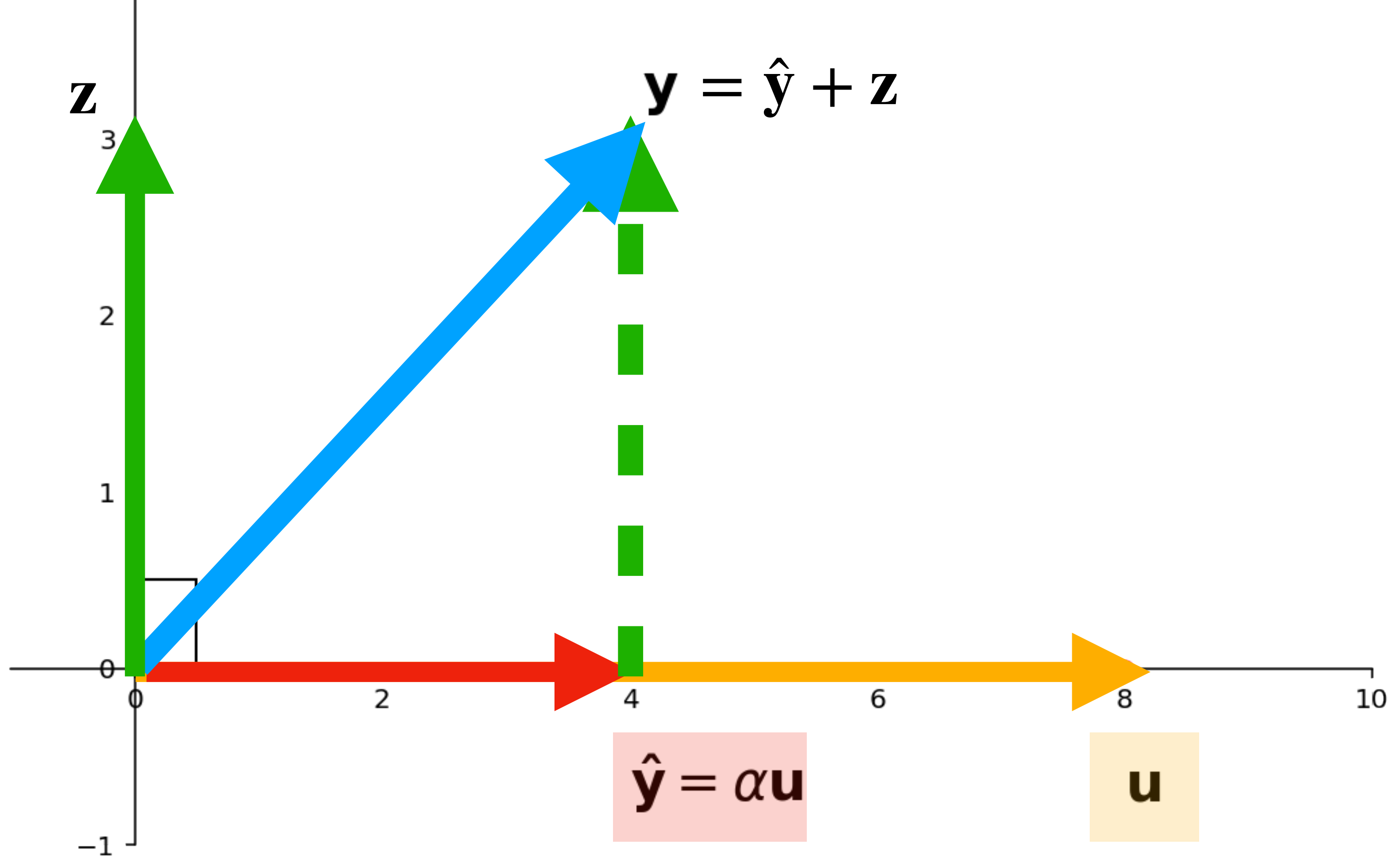

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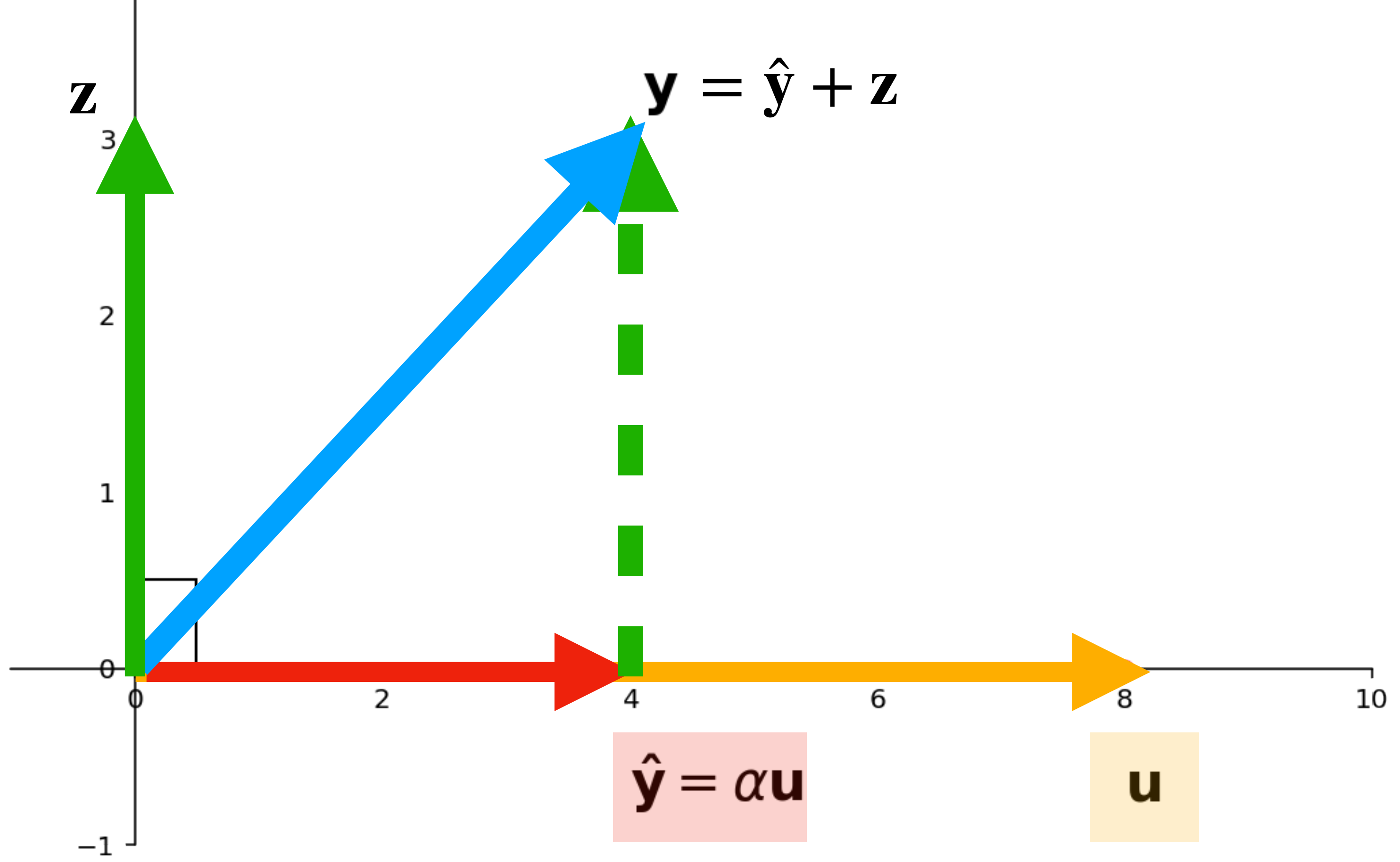
Therefore:

$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

What we know



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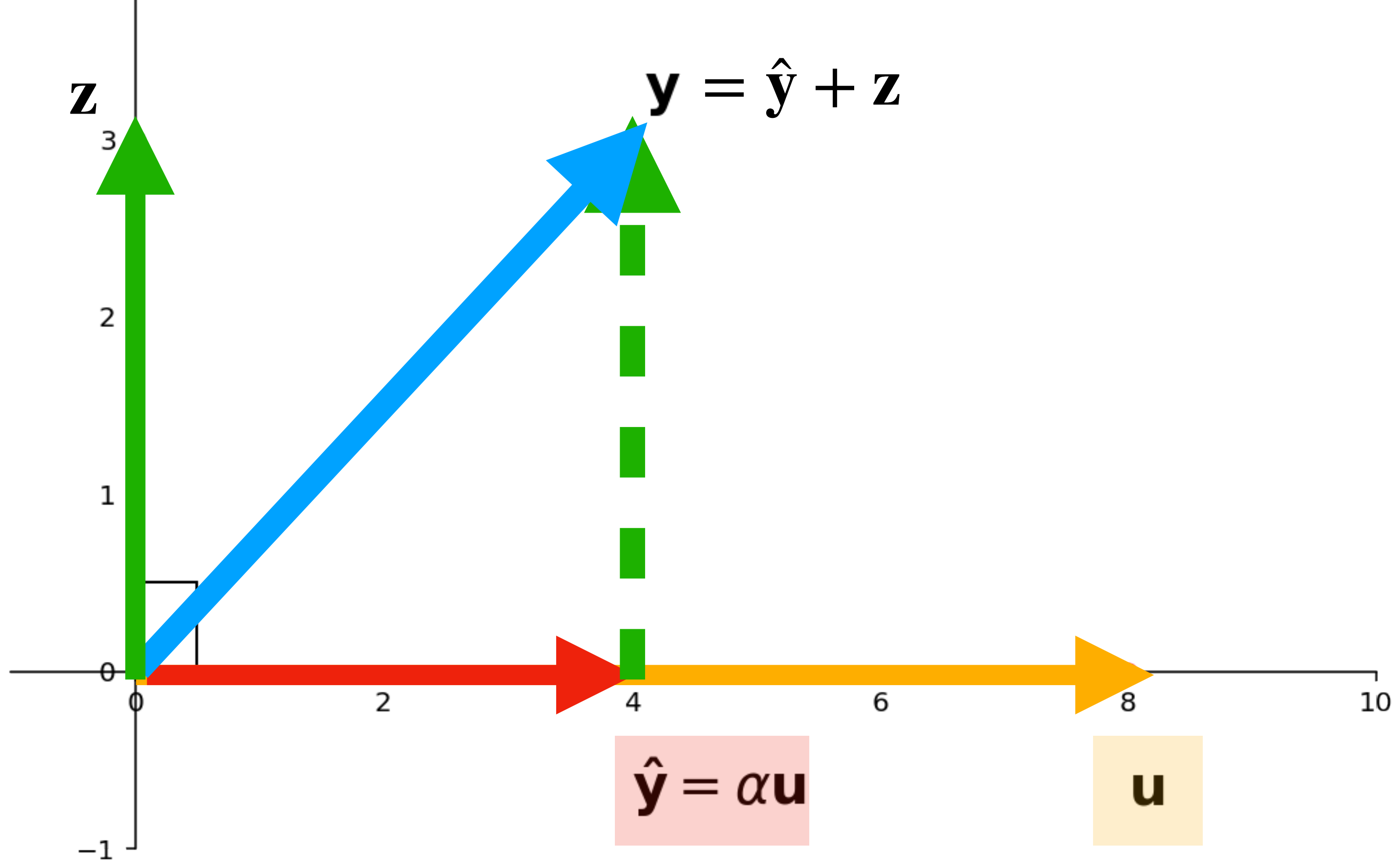
Once we have α, we can compute both $\hat{\mathbf{y}}$ and $\mathbf{z}$

Step 1: Finding α



$$\langle \mathbf{y} - \alpha \mathbf{u}, \mathbf{u} \rangle = 0$$

Let's solve for α, $\hat{\mathbf{y}}$ and $\mathbf{z}$:

$$\langle \vec{y} - \alpha \vec{u}, \vec{u} \rangle = \langle \vec{y}, \vec{u} \rangle - \alpha \langle \vec{u}, \vec{u} \rangle = 0$$

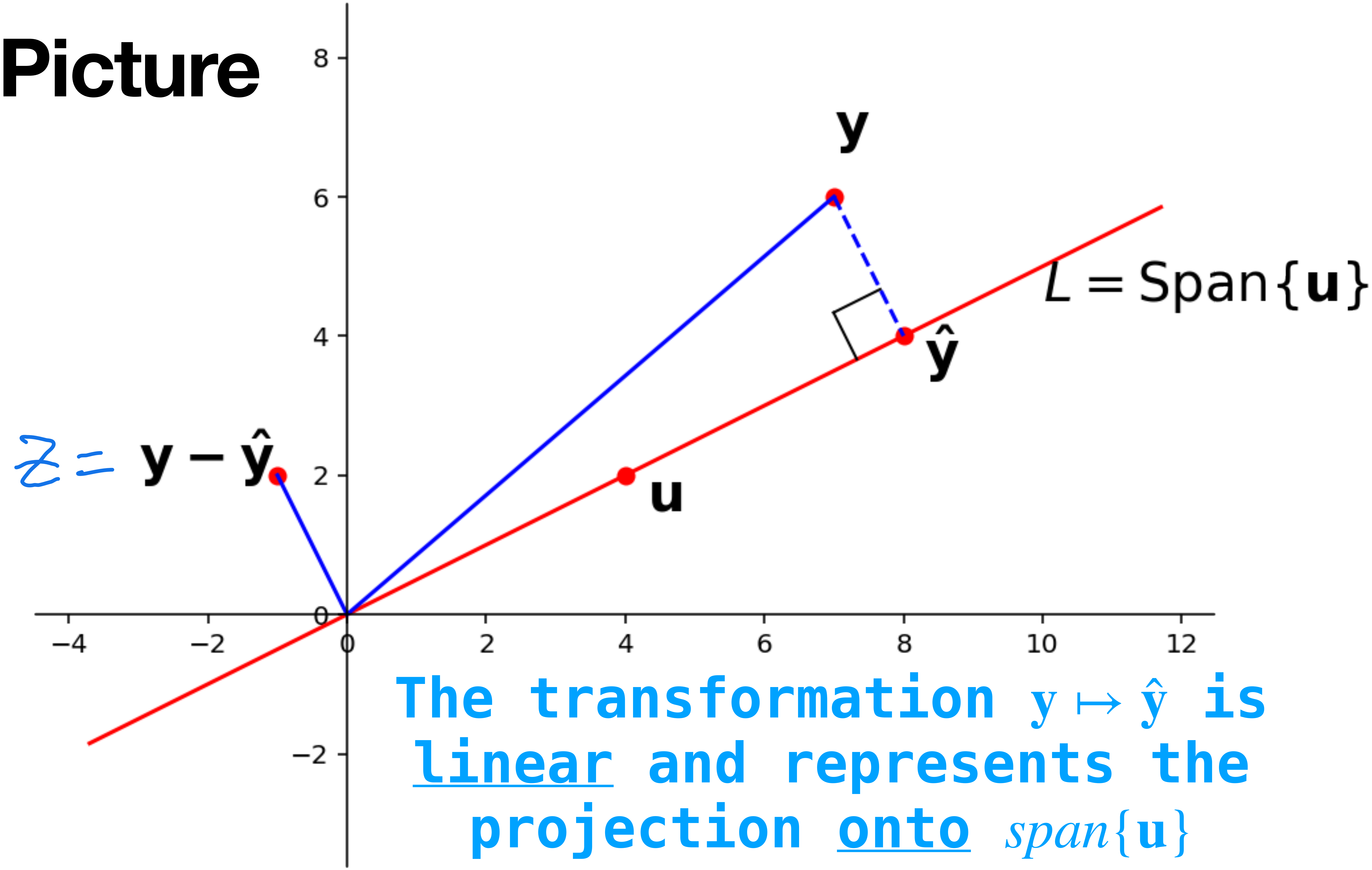
$$\alpha \langle \vec{u}, \vec{u} \rangle = \langle \vec{y}, \vec{u} \rangle$$

$$\boxed{\alpha = \frac{\vec{y}^T \vec{u}}{\vec{u}^T \vec{u}}}$$

$$\hat{y} = \alpha \vec{u}$$

$$\vec{z} = \vec{y} - \hat{y} = \vec{y} - \alpha \vec{u}$$

The Picture



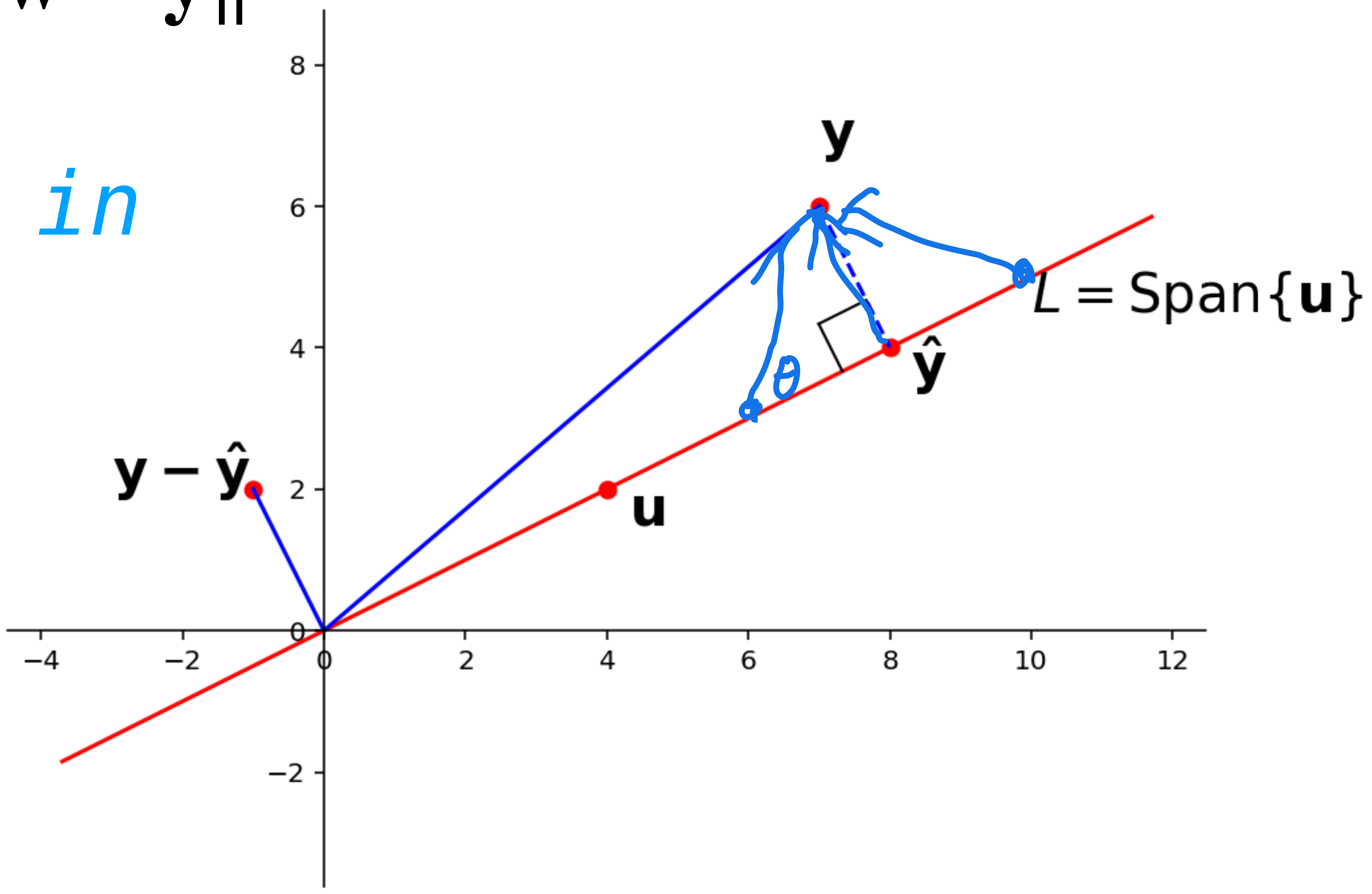
The transformation $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is <u>linear</u> and represents the projection <u>onto</u> $span\{\mathbf{u}\}$

$\hat{\mathbf{y}}$ and Distance

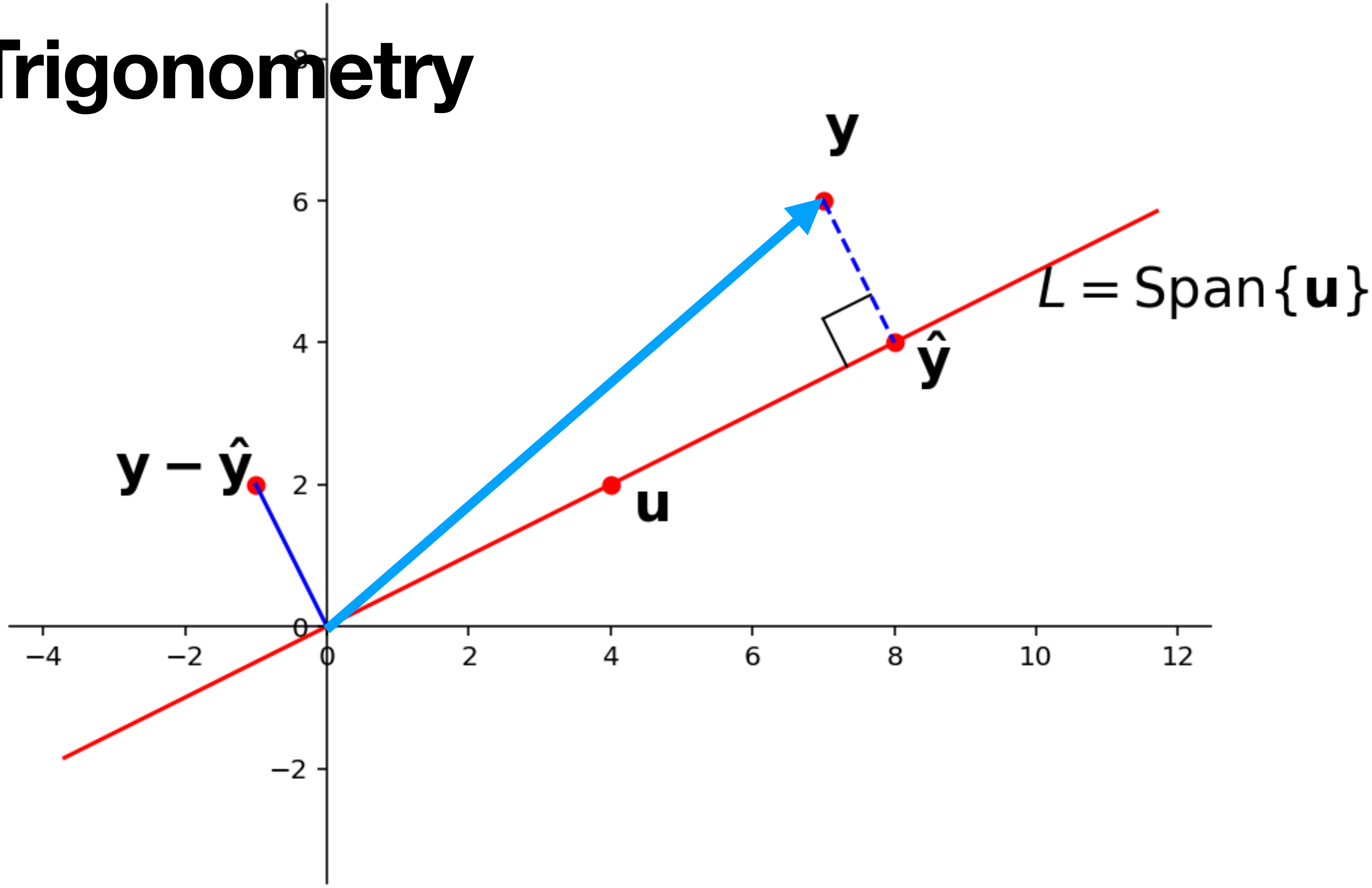
Theorem. $\|\hat{\mathbf{y}} - \mathbf{y}\| = \min_{\mathbf{w} \in span\{\mathbf{u}\}} \|\mathbf{w} - \mathbf{y}\|$

$\hat{\mathbf{y}}$ is the <u>closest</u> vector in $span\{\mathbf{u}\}$ to $\mathbf{y}$.

"Proof" by inspection:



The Trigonometry



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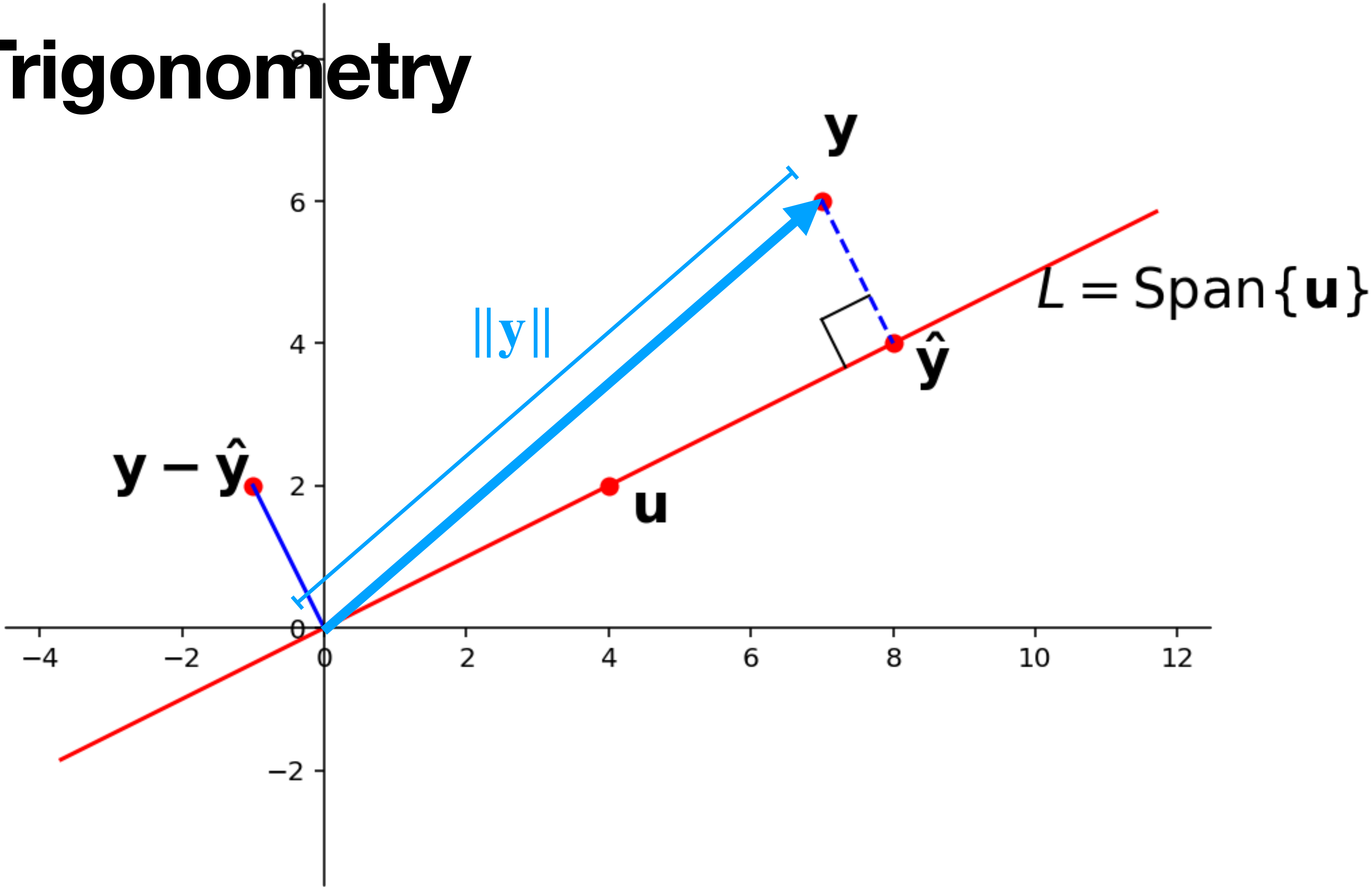
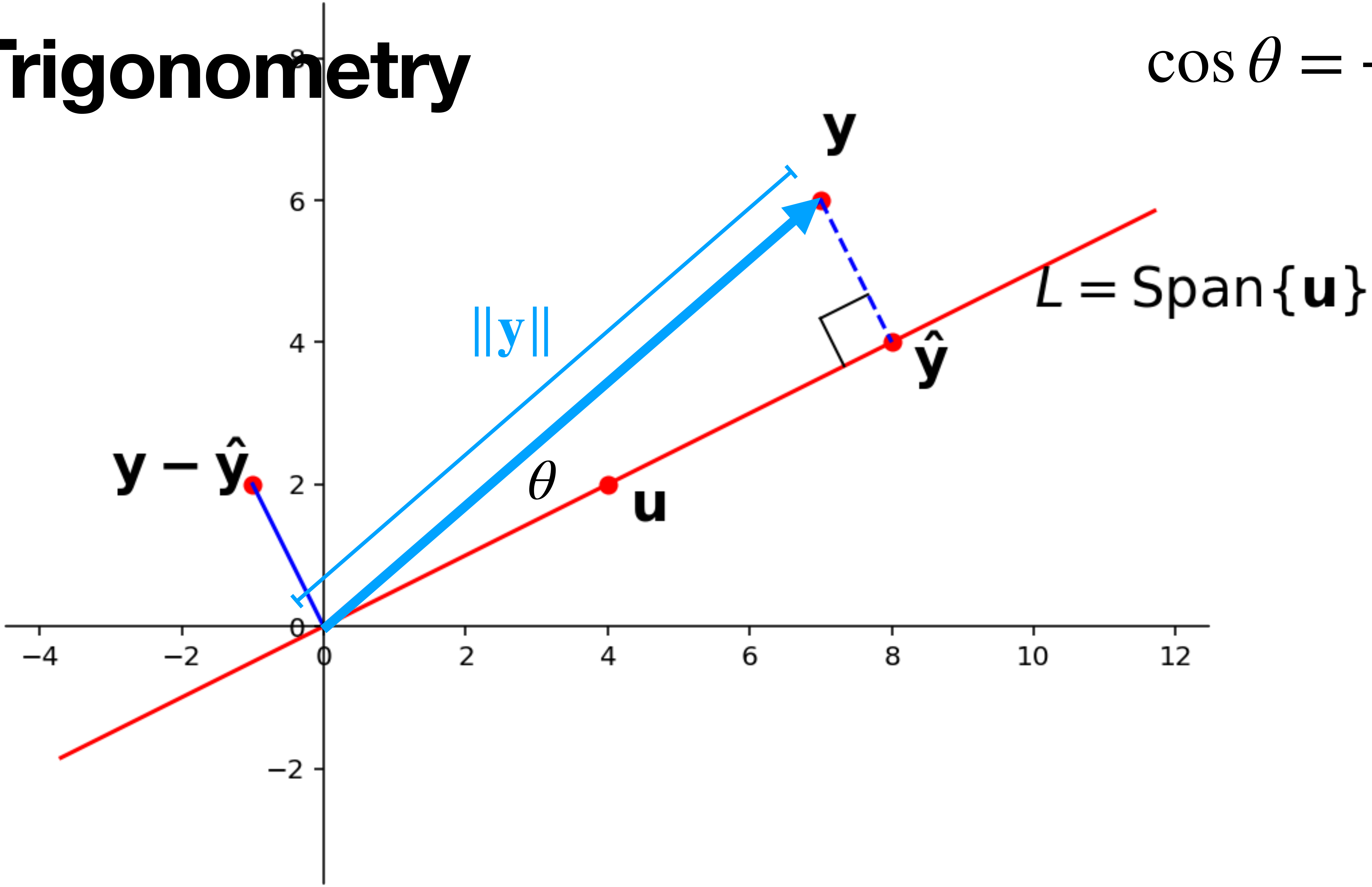


The Trigonometry

$$\cos\theta = \frac{\|\hat{\mathbf{y}}\|}{\|\mathbf{y}\|}$$



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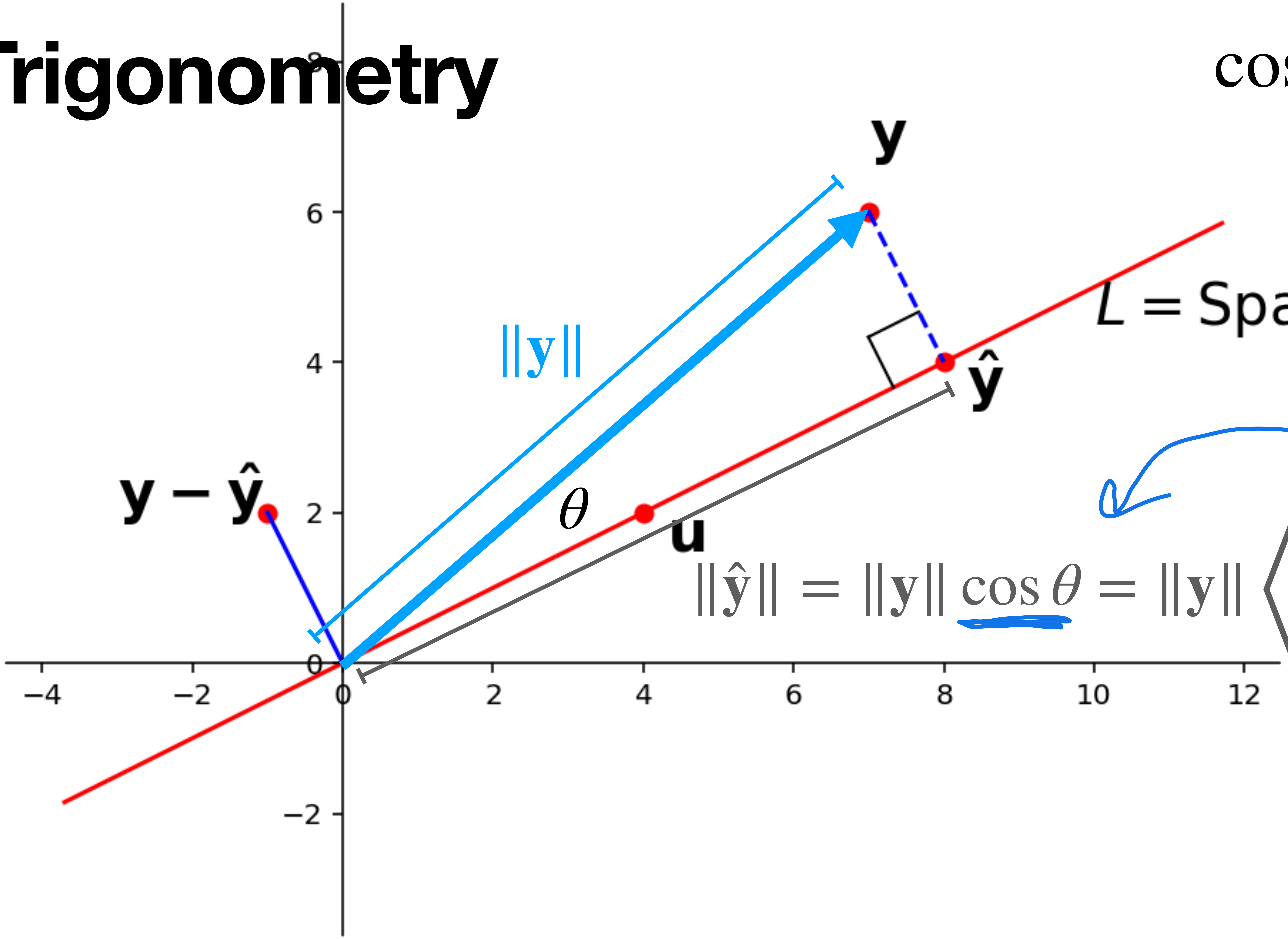

$$\|\hat{\mathbf{y}}\| = \|\mathbf{y}\|\cos\theta = \|\mathbf{y}\|\left\langle \frac{\mathbf{y}}{\|\mathbf{y}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle$$

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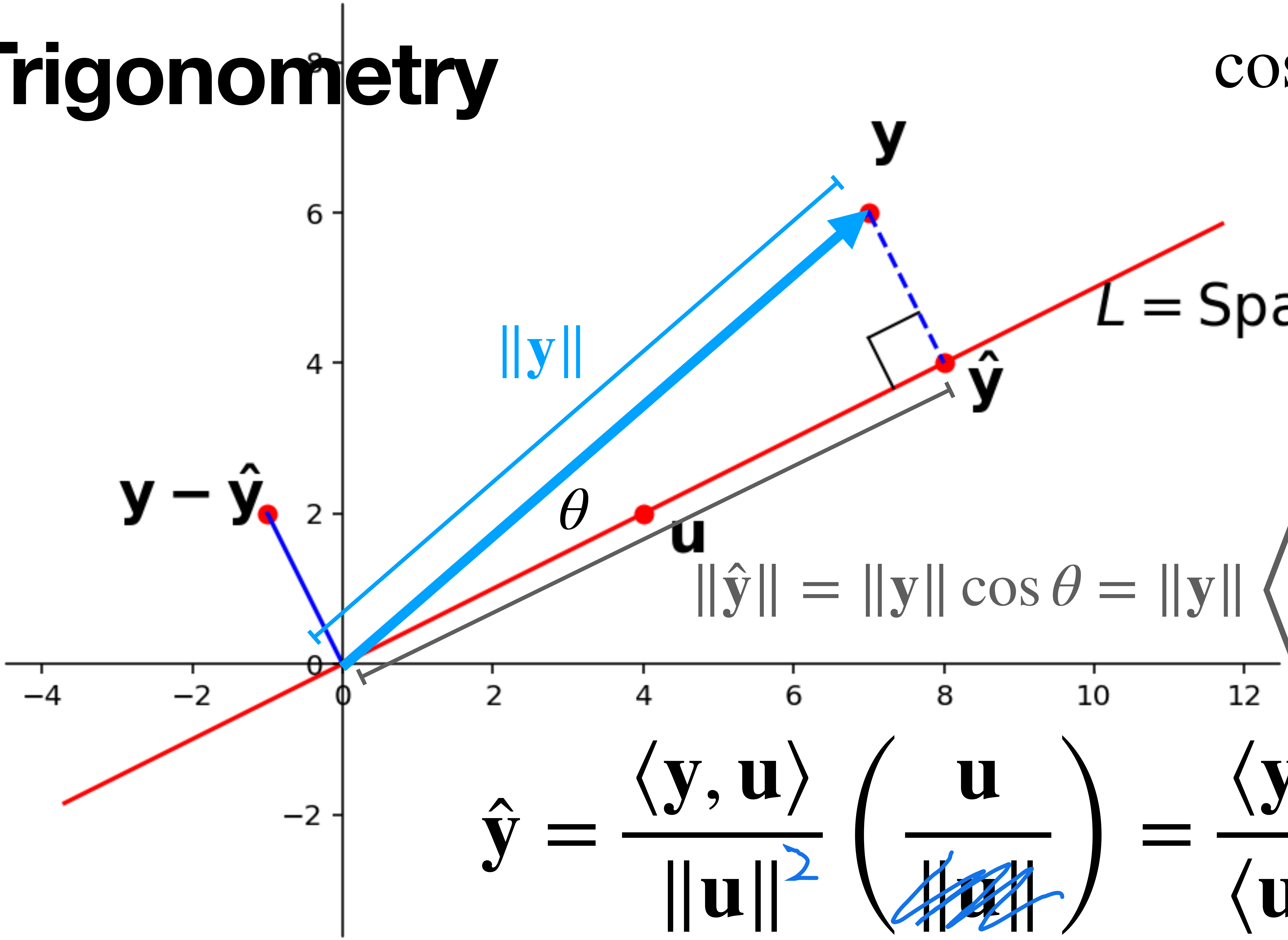

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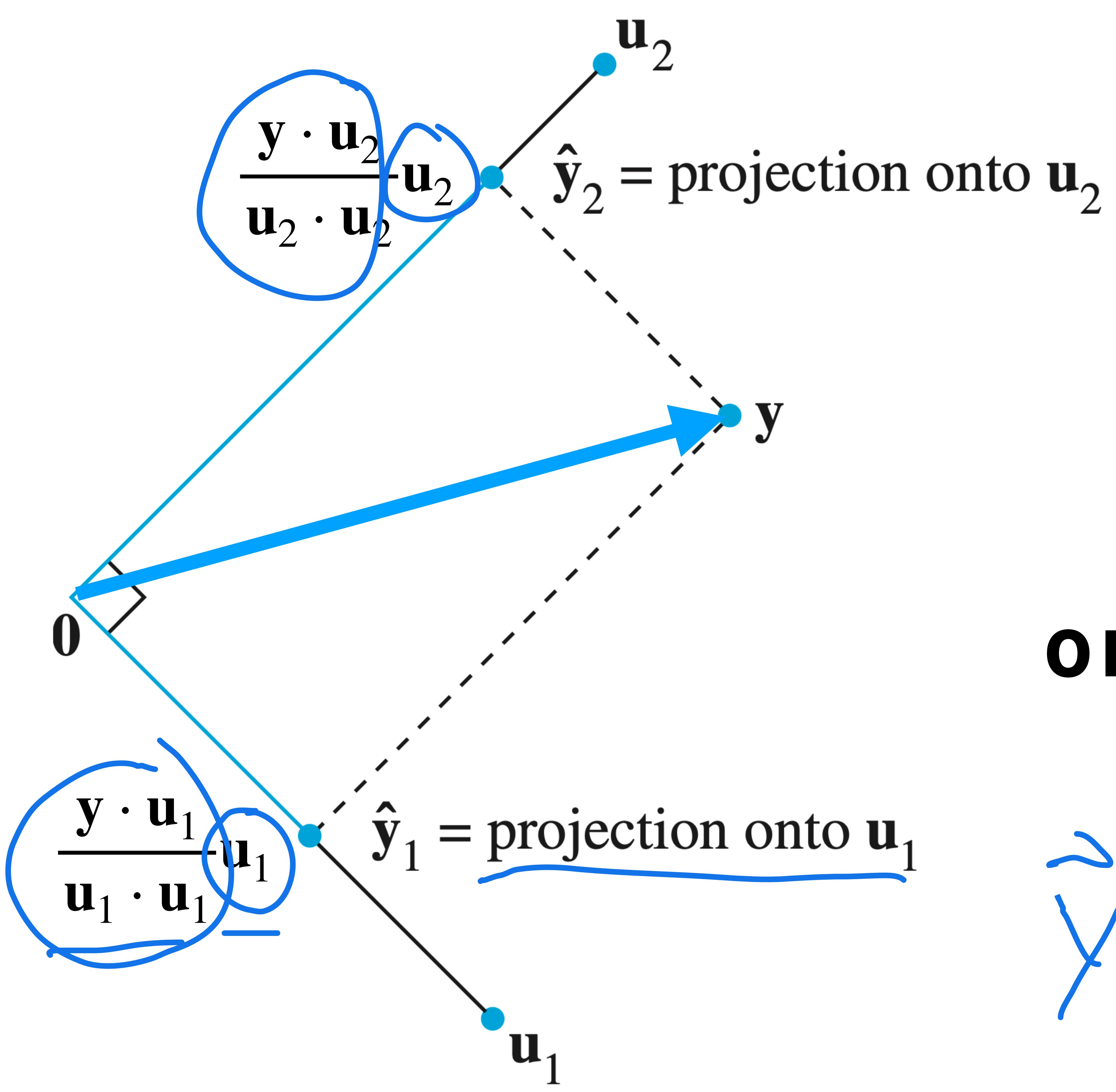
$$\hat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2}\left(\frac{\mathbf{u}}{\cancel{\|\mathbf{u}\|}}\right) = \frac{\langle \mathbf{y}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$$

Orthogonal Projections and Orthogonal Bases



Each <u>component</u> of $\mathbf{y}$ written in terms of an *orthogonal* basis is an **orthogonal projection onto to a basis vector**

$\tilde{y} = \hat{y}_1 + \hat{y}_2$

How To:

Question. Find the projection of $\mathbf{y}$ onto the span of $\mathbf{u}$

Solution. Calculate $\alpha = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$, then the solution is $\boxed{\alpha\mathbf{u}} = \hat{y}$

$\|\vec{u}\|^2$

Question

$$\alpha = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$$

Find the matrix which implements orthogonal projection onto the span of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \vec{u}$

$$\vec{y} \mapsto \alpha\vec{u} = \vec{u}\left(\frac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}\right) = \vec{u}\alpha$$

$$= \vec{u}\left(\frac{\vec{u}^T\vec{y}}{\|\vec{u}\|^2}\right) = \frac{1}{\|\vec{u}\|^2}\left(\vec{u}\vec{u}^T\right)\vec{y}$$

$$\underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\vec{u}} \frac{\overbrace{\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}}^{\vec{u}^T}}{\underbrace{1+1+4}_{\|u\|^2}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

Answer

$$\frac{1}{6\cancel{\sqrt{5}}}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

Orthonormal Sets

Orthogonal sets would be easier to work with if every vector was a unit vector

Orthonormality

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Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of <u>unit</u> vectors

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Orthonormal Matrices

$m \times n$

Definition. A matrix is **orthonormal** if its columns form an orthonormal set

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This is incredibly confusing, but we'll try to be consistent and clear

Orthonormal Matrices and Transposition

Theorem. For an $m \times n$ orthonormal matrix U

$$\underbrace{U^T}_{(n\times m)}\underbrace{U}_{(m\times n)} = I_n$$

Verify:

$$\begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} & & \\ & \vec{u}_i^T \vec{u}_j & \\ & & \end{bmatrix}$$

0 for $i \neq j$

if $i = j$, $\|u_i\|^2$ $= 1$

Inverses of Orthogonal Matrices

Theorem. If an $n \times n$ matrix U is orthogonal (square orthonormal) then it is invertible and

$$U^{-1} = U^T$$

Verify: By prior argument

$U^T U = \mathrm{Id}$

also inverse

Orthonormal Matrices and Inner Products

Theorem. For a $m \times n$ orthonormal matrix U, and any vectors x and y in R^n

$$\langle \underbrace{Ux, Uy}_{\text{outputs}} \rangle = \langle \underbrace{x, y}_{\text{inputs}} \rangle$$

$\mathbb{R}^m$ $\mathbb{R}^n$

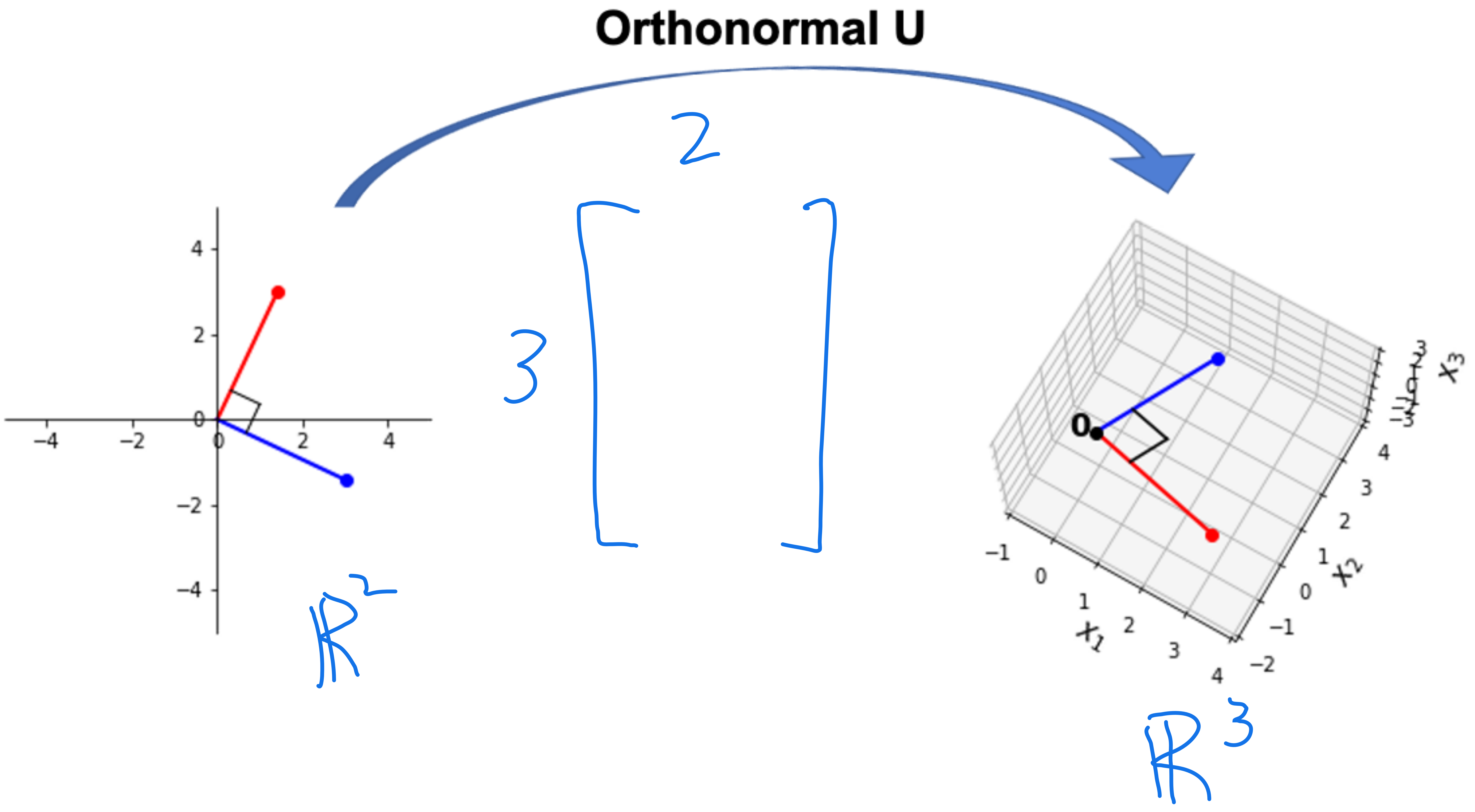
Orthonormal matrices preserve inner products

Verify: $(U\vec{x})^T(U\vec{y}) = \vec{x}^T \underbrace{U^T U}_{I_n} \vec{y} = \vec{x}^T\vec{y}$

Length, Angle, Orthogonality Preservation

Since lengths and angles are defined in terms of inner products, they are also preserved by orthonormal matrices:

The Picture



Example

~~[illegible]~~ Verify $\|Ux\|^2 = \|x\|^2$

$$U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \qquad x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

$$\|x\|^2 = 2 + 9 = 11$$

$$U\vec{x} = \begin{bmatrix} 1+2 \\ 1-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|U\vec{x}\|^2 = 9 + 1 + 1 = 11$$

length/norm preserved!

Question (Conceptual)

Suppose A is an $m \times n$ matrix with orthogonal but ***not*** *orthonormal columns. What is A^TA?*

col vectors NOT unit norm

$$\underbrace{\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}}_{A} = \begin{bmatrix} \vec{a}_i^T \vec{a}_j \end{bmatrix}$$

if $i \neq j$: 0

if $i = j$: $\|\vec{a}_i\|^2$

Answer

If $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$ then $A^T A$ is a diagonal matrix D where

$$D_{ii} = \|\mathbf{a}_i\|^2$$

Summary

Orthogonal sets allow for simpler calculations of coordinates

Finding these coordinates is a really about find the orthogonal projections onto each vector in the orthogonal set

We can apply these ideas to matrices and describe a class of very well behaved transformations via orthonormal matrices