Problem 1 Solution

$$A = \begin{array}{c} \left[\begin{array}{cccc|c} 0 & 2 & 0 & 4 & 1 \\ 0 & -3 & 1 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \\ \begin{array}{ccccc} \overset{\shortparallel}{\vec{a}_1} & \overset{\shortparallel}{\vec{a}_2} & \overset{\shortparallel}{\vec{a}_3} & \overset{\shortparallel}{\vec{a}_4} & \overset{\shortparallel}{\vec{a}_5} \end{array} \end{array}$$

(A) Write down a basis for Col A.

$$A \sim \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow \quad \uparrow \quad \uparrow$

pivot columns

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(B) Write down a basis for Nul A.

From RREF:

x_1 free

$x_2 = -2x_4$

$x_3 = -3x_4$

x_4 free

$x_5 = 0$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$\uparrow \quad \nearrow$

basis elements

(C) Write down a nontrivial linear dependence relation between the columns of A.

$c_1\vec{a}_1 + \cdots + c_5\vec{a}_5 = 0$

$c_1 = 1$

From RREF $\boxed{\vec{a}_4 = 2\vec{a}_2 + 3\vec{a}_3}$

(D) If interpreted as the augmented matrix of a nonhomogeneous linear system, ~~[illegible]~~ how many solutions does the system have?

None, inconsistent due to 3rd row in RREF

(E) Does this system have a least squares solution? If so, is it unique?

Yes, all systems have a least squares solution.
No, it would not be unique as $\begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \\ | & | & | & | \end{bmatrix}$ has a nontrivial null space. (Actually, it's 2-dimensional, so there'd be a two-dimensional space of least squares solutions).

Problem 2 Sol'n

A is a 7×5 matrix.

$$A^TA = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(A) What are the singular values of A?

$$\det(A^TA - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 & 0 & 0 & 0 \\ -1 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \right)$$

$$\begin{pmatrix}\text{cofactor} \\ \text{expansion}\end{pmatrix} = (-\lambda)^2(4-\lambda)\left((2-\lambda)^2 - 1\right)$$

$$= \lambda^2(4-\lambda)\underbrace{(\lambda^2 - 4\lambda + 4 - 1)}_{\lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)}$$

$$= -\lambda^2(\lambda-4)(\lambda-1)(\lambda-3)$$

nonzero eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 1$

$\boxed{\text{sing. values: } \sigma_1 = 2, \quad \sigma_2 = \sqrt{3}, \quad \sigma_3 = 1}$

(B) Non cofactor way:

$$E_{R_2 \leftarrow (2-\lambda)R_2}(A^TA - \lambda I) = \begin{bmatrix} 2-\lambda & -1 & 0 & 0 & 0 \\ -(2-\lambda) & (2-\lambda)^2 & 0 & 0 & 0 \\ 0 & 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$$\sim \begin{bmatrix} 2-\lambda & -1 & 0 & 0 & 0 \\ 0 & (2-\lambda)^2-1 & 0 & 0 & 0 \\ 0 & 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$E_{R_1 \leftarrow R_1 + R_2}$

$$\det\left(E_{R_2 \leftarrow (2-\lambda)R_2}(A^TA - \lambda I)\right) = (-\lambda)^2(4-\lambda)(2-\lambda)\left((2-\lambda)^2 - 1\right)$$

$\det(E_{R_1 \leftarrow R_1 + R_2})$ $\quad \parallel$

$$\det(E_{R_2 \leftarrow (2-\lambda)R_2})\det(A^TA - \lambda I)$$

$\parallel$ $\quad \parallel$

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$$\cancel{(2-\lambda)}\det(A^TA - \lambda I) = (\lambda)^2(4-\lambda)\cancel{(2-\lambda)}(\lambda^2 - 4\lambda + 3)$$

divide out

rest is same...

(B) What is nullity $A = \dim \operatorname{Nul} A$? Use the fact proven in lecture that $\operatorname{Nul} A = \operatorname{Nul} A^TA$.

As A^TA is symmetric, it's multiplicity bounds are achieved. The power of λ is 2 in the char. polynomial, so $\dim \operatorname{Nul} A^TA = 2$. Thus $\boxed{\dim \operatorname{Nul} A = \text{nullity } A = 2}$

(C) What is $\operatorname{rk} A = \dim \operatorname{Col} A$?

By Rank-Nullity: $\operatorname{rk} A + \text{nullity } A = 5 \Rightarrow \operatorname{rk} A = 3$

(D) What is $\operatorname{rk}(A^T)$ & nullity A^T? (Hint: row operations preserve the row space of A)

$\operatorname{Rk} A^T = \dim \operatorname{Col} A^T = \dim \operatorname{Row} A = \#$ pivot ~~columns~~ rows in RREF
$= \operatorname{rk} A = 3$

$\operatorname{rk} A^T + \text{nullity } A^T = 7 \Rightarrow \text{nullity } A^T = 4$

(E) Write down V^T in the SVD of A, assuming ordering of singular values from greatest to least

We need the normalized eigenvectors of A^TA.

$\lambda_1 = 4 \qquad \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ by inspection

$\lambda_2 = 3$

$$A^TA - 3I = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_3 = 1$

$$A^TA - I = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_4 = \lambda_5 = 0$

$$\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{by inspection}$$

$$V^T = \begin{bmatrix} \text{——} \vec{v}_1^T \text{——} \\ \text{——} \vec{v}_2^T \text{——} \\ \text{——} \vec{v}_3^T \text{——} \\ \text{——} \vec{v}_4^T \text{——} \\ \text{——} \vec{v}_5^T \text{——} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3 Solution

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 3 \\ 7 \end{bmatrix}}_{=\vec{b}_1}, \underbrace{\begin{bmatrix} -2 \\ -4 \end{bmatrix}}_{=\vec{b}_2} \right\}$$

(A) Write down a matrix that implements the change of basis from the standard basis to $\mathcal{B}$.

$$\begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix}^{-1} = \frac{1}{14-12} \begin{bmatrix} -4 & 2 \\ -7 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 1 \\ -7/2 & 3/2 \end{bmatrix}}$$

(Recall that $\begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix} [\vec{x}]_{\mathcal{B}} = [\vec{x}]$)

in standard basis

(B) Construct ~~a basis~~ an orthonormal basis $\mathcal{C}$ that contains a scalar multiple of $\vec{b}_2$.

$\vec{b}_2$ prop. to $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

orthogonal to this is $\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

$$\mathcal{C} = \left\{ \underbrace{\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}}_{=\vec{c}_1}, \underbrace{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}_{=\vec{c}_2} \right\}$$

(one of a few choices)

(C) Write down the 2×2 matrix (in standard basis) that implements the linear transformation that maps:

$$\vec{c}_1 \mapsto 3\vec{c}_1$$
$$\vec{c}_2 \mapsto -\vec{c}_2$$

$$P \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} P^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

COB matrix that maps to $\mathcal{C}$

$$= \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 6/\sqrt{5} & -3/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 12-1 & -6-2 \\ -6-2 & 3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 11/5 & -8/5 \\ -8/5 & -1/5 \end{bmatrix}$$

(Note some ambiguity here also depending on choice & ordering of $\vec{c}_1$ & $\vec{c}_2$)