Final Exam

CAS CS 132: Geometric Algorithms

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- ▷ You will have approximately 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Please do not remove any pages from the exam.
- ▷ Please put your **final** solution in the solution box *and nothing else*. **You should do your work outside of the box!**
- ▷ You must show your work on all problems unless otherwise specified. A solution without work will be considered incorrect (and will be investigated for potential academic dishonesty).
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

1 Column Space and Null Space

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

A. (3 points) Determine a basis for $\mathsf{Nul}\, A$.

Solution.

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x$

x_2 is free

$x_3 = 0$

$x_4 = 0$

(or just note that last three columns are L.I.)

B.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(3 points) Determine a linear equation over the variables x_1, x_2, x_3, x_4 whose solution set is $\mathsf{Col}\, A$ (that is, (x_1, x_2, x_3, x_4) satisfies the linear equation if and only if the vector $[x_1\ x_2\ x_3\ x_4]^T$ is in the column space of A).

Solution.

$$x_3 - 2x_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 & b_1 \\ 1 & 2 & 4 & -4 & b_2 \\ 0 & 0 & 2 & 4 & b_3 \\ 0 & 0 & 1 & 2 & b_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 & b_1 \\ 1 & 2 & 4 & -4 & b_2 \\ 0 & 0 & 1 & 2 & b_3/2 \\ 0 & 0 & 1 & 2 & b_4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 & b_1 \\ 1 & 2 & 4 & -4 & b_2 \\ 0 & 0 & 1 & 2 & b_3/2 \\ 0 & 0 & 0 & 0 & b_4 - b_3/2 \end{bmatrix}$$

(or any $\underline{\text{scaling}}$ of the above equation)

2 Linear Models

$$\{(-1, 1), (0, -1), (1, 1), (2, 3), (4, 5)\}$$

(3 points) Determine the equations for finding the best-fit curve of the form

$$\beta_0 + \beta_1 x^3 + \beta_2(2^x)$$

(where β_0, β_1, and β_2 are parameters) for the above data using least-squares regression. That is, determine the design matrix X and vector of observations $\mathbf{y}$ such that the least-squares solution of

$$X \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{y}$$

determines the parameters for best-fit curve.

Solution.

$$\underbrace{\begin{bmatrix} 1 & -1 & 1/2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 8 & 4 \\ 1 & 64 & 16 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\vec{\beta}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \\ 5 \end{bmatrix}}_{\vec{y}}$$

3 Exponentials of Matrices

In a homework problem we saw that if A is diagonalizable (i.e., it can be expressed as PDP^{-1} for a diagonal matrix D) then we can define $A^{1/2}$ as $PD^{1/2}P^{-1}$, where

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}^{1/2} = \begin{bmatrix} d_1^{1/2} & 0 & \dots & 0 & 0 \\ 0 & d_2^{1/2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1}^{1/2} & 0 \\ 0 & 0 & \dots & 0 & d_n^{1/2} \end{bmatrix}$$

That is, we can take the square root of each entry along the diagonal of D in the diagonalization of A. It turns out that for any function $f : \mathbb{R} \to \mathbb{R}$ and any diagonalizable matrix A as above, we can define $f(A)$ as $Pf(D)P^{-1}$ where

$$f\left(\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}\right) = \begin{bmatrix} f(d_1) & 0 & \dots & 0 & 0 \\ 0 & f(d_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & f(d_{n-1}) & 0 \\ 0 & 0 & \dots & 0 & f(d_n) \end{bmatrix}$$

That is, we apply f to each entry along the diagonal of D in the diagonalization of A. We will use this fact to reason about exponentials of matrices. *(Problem continued on next page.)*[1]

[1] Credit to Vishesh Jain and Abhinit Sati for suggesting a version of this problem.

A.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(4 points) Determine a diagonalization of A. You final solution should be in the form PDP^{-1}, where D is a diagonal matrix and P^{-1} is an explicit matrix, i.e., you must compute the inverse of P.

Solution.

$$\underset{P}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}}$$

$$\det(A - \lambda I) = (\lambda + 2)(\lambda - 4) + 8 = \lambda^2 - 2\lambda - 8 + 8$$

$$\lambda = 2, 0$$

$$A - 2I = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

(Problem 2A Continued)

B.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(2 points) Determine the matrix 2^A. Your final solution should be an explicit matrix, i.e, you must compute all matrix multiplications. (*Hint.* Use $f(x) = 2^x$.)

Solution.

$$\begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$2^A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$$

C.

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 \\ 8 & -8 \end{bmatrix}$$

(4 points) Determine the characteristic polynomial of $2^A 2^B$. Give your solution in expanded form, i.e., not factored. (*Hint.* Don't try to compute this matrix product directly. Use properties of exponentiation).

Solution.

$\lambda^2 - 12\lambda + 32$

$$2^A 2^B = 2^{A+B}$$

$$A + B = \begin{bmatrix} 8 & -5 \\ 6 & -3 \end{bmatrix}$$

$$\det(A + B - \lambda I) = (\lambda - 8)(\lambda + 3) + 30$$
$$= \lambda^2 - 5\lambda - 24 + 30$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda - 2)(\lambda - 3)$$

$$\det(2^{A+B} - \lambda I) = (\lambda - 2^2)(\lambda - 2^3)$$
$$= (\lambda - 4)(\lambda - 8)$$
$$= \lambda^2 - 12\lambda + 32$$

4 True/False Questions

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.[2]

A. (1 point) For any matrix A with orthogonal columns, $A^T A = I$.

○ True

● False

B. (1 point) If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.

● True

○ False

C. (1 point) For any matrix A, if A is orthogonally diagonalizable, then so is A^T.

● True

○ False

D. (1 point) For any vector $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^n$, $\mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathsf{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$.

○ True

● False

E. (1 point) For any matrix A and vector $\mathbf{v}$, if $\mathbf{v}$ is an eigenvector of the matrix A then it is also an eigenvector of the matrix A^2.

● True

○ False

F. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{v} \in \mathbb{R}^n$, if $\|A\mathbf{v}\| = \mathbf{0}$ then $\mathbf{v} \in \mathsf{Nul}\, A$.

● True

○ False

G. (1 point) For any matrix A, we have $\mathsf{rank}(A) = \mathsf{rank}(A^T)$.

● True

○ False

H. (1 point) For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\mathbf{v}_3 \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

○ True

● False

I. (1 point) For any square matrices A and B, we have $\det(AB^T) = \det(A)\det(B)$

● True

○ False

J. (1 point) For any quadratic form $Q(\mathbf{x})$, the vector $\mathsf{argmax}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$ is unique.

○ True

● False

[2] Credit to Vishesh Jain and Abhinit Sati for suggesting some parts of this question.

K. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, then $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

○ True

● False

L. (1 point) If A is a square matrix with strictly positive entries, then there is diagonal matrix D such that AD is stochastic.

● True

○ False

5 Singular Value Decomposition

A.

$$A = \begin{bmatrix} 2 & -4 & 4 \\ 2 & 2 & 1 \\ -2 & 4 & -4 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 36 & 0 & -36 \\ 0 & 9 & 0 \\ -36 & 0 & 36 \end{bmatrix}$$

(4 points) Determine a *reduced* singular value decomposition of A, given that $6\sqrt{2}$ and 3 are its nonzero singular values and $\mathsf{rank}(A) = 2$. (*Reminder.* If $U\Sigma V^T$ is an SVD of A and $\mathsf{rank}(A) = r$, then we can get a reduced SVD of A by making Σ an $r \times r$ diagonal matrix and dropping columns of U and V so that the multiplication $U\Sigma V^T$ is well-defined.)

Solution.

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ -1/\sqrt{2} & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = U\Sigma V^T$$

$$\Sigma = \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 3 \end{bmatrix}$$

$$AA^T - 72I = \begin{bmatrix} -36 & 0 & -36 \\ 0 & \square & 0 \\ -36 & 0 & -36 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$AA^T - 9I = \begin{bmatrix} \square & 0 & \square \\ \square & 0 & \square \\ \square & 0 & \square \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T v_1 = \frac{4}{\sqrt{2}} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

(these calculations are for A^T)

$$A^T v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

(Problem 5A Continued)

B.

$$\mathbf{b} = \begin{bmatrix} 10 \\ 18 \\ -14 \end{bmatrix}$$

(3 points) Let A be the matrix from the previous part. Determine the length of the *shortest* least-squares solution to the equation $A\mathbf{x} = \mathbf{b}$. (*Hint.* Use the pseudoinverse of A, i.e., $A^+ = V\Sigma^{-1}U^T$, and keep in mind that matrices with orthonormal columns preserve lengths.)

Solution.

$\sqrt{40}$

$$\|A^+\vec{b}\| = \|V\Sigma^{-1}U^T\vec{b}\| = \|\Sigma^{-1}U^T\vec{b}\|$$

$$= \left\| \begin{bmatrix} \frac{1}{6\sqrt{2}} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 18 \\ -14 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} \frac{1}{6\sqrt{2}} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{24}{\sqrt{2}} \\ 18 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right\| = \sqrt{4+36} = \sqrt{40}$$

(Problem 5B Continued)

6 Dependence Relations

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$$

(5 points) Determine a dependence relation for the above vectors. That is, write **0** as a linear combination of the above vectors.

Solution.

$$\vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 + \vec{v}_4 = \vec{0}$$

$$\begin{bmatrix} 1 & -1 & -3 & -2 \\ 0 & 2 & 2 & -2 \\ 1^{-1} & 0^{+1} & 4^{+3} & 3^{+2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 1^{-1} & 7^{-1} & 5^{+1} \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -3^{+3} & -2^{+3} \\ 0 & 1 & 1^{-1} & -1^{-1} \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1^{+1} & 0 & 1^{-2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_1 = x_4$

$x_2 = 2x_4$

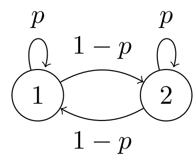
$x_3 = -x_4$

x_4 is free

(Problem 6 Continued)

7 Stochastic Matrices

A. Consider the following state diagram.



(2 points) Write the transition matrix T for the above diagram in terms of p. In the following parts, T will refer to this matrix.

Solution.

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

B. (3 points) Given $0 < p < 1$, determine $\lim_{k\to\infty} T^k \mathbf{e}_1$ (where $\mathbf{e}_1$ is the first standard basis vector). Justify your answer.

Solution.

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Justification.

The long term behavior of a Markov chain is given by the (probability) eigenvector for $\lambda_1 = 1$, i.e., the steady-state.

$$T - I = \begin{bmatrix} p-1 & 1-p \\ 1-p & p-1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

C. (4 points) Determine λ_2, the *second* largest eigenvalue of T, and a corresponding eigenvector. (*Hint.* Keep in mind that T is symmetric and, hence, orthogonally diagonalizable.)[3]

Solution.

$\lambda_2 = 2p - 1$ $\qquad$ $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\det(T - \lambda I) = (\lambda - p)^2 - (1-p)$$

$$= \lambda^2 - 2p \cancel{+ p^2} - 1 + 2p \cancel{- p^2}$$

$$= (\lambda - 1)(\lambda - (2p-1))$$

$$\lambda = 1,\ 2p-1$$

(v_2 must be (a multiple of) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

since T has an orthogonal eigenbasis)

[3]The second largest eigenvalue tells us about the *rate of convergence* to the steady-state distribution. The smaller $|\lambda_2|$, the faster the convergence to the steady-state distribution.

(Problem 7C Continued)

D. (4 points) Give a closed-form solution for $T^k \begin{bmatrix} 1-q \\ q \end{bmatrix}$ in terms of p, q and k (*Hint.* Again, keep in mind that A is symmetric and, hence, orthogonally diagonalizable).

Solution.

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + (2q-1)(2p-1)^k \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1-q \\ q \end{bmatrix} = \frac{\left\langle \begin{bmatrix} 1-q \\ q \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\left\langle \begin{bmatrix} 1-q \\ q \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(2q-1)}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T^k \begin{bmatrix} 1-q \\ q \end{bmatrix} = 1^k \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + (2q-1)(2p-1)^k \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + (2q-1)(2p-1)^k \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

(there are a couple equivalent forms)

(Problem 7D Continued)

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