

Practice Final Exam

CAS CS 132: Geometric Algorithms

Name:

BUID:

- ▷ You will have approximately 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Please do not remove any pages from the exam.
- ▷ Please put your **final** solution in the solution box *and nothing else*. **You should do your work outside of the box!**
- ▷ You must show your work on all problems unless otherwise specified. A solution without work will be considered incorrect (and will be investigated for potential academic dishonesty).
- ▷ We will not look at any work on the pages marked “*This page is intentionally left blank.*” You should use these pages for scratch work.

1 Column Space and Null Space

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

A. (3 points) Determine a basis for $\text{Nul } A$.

Solution.

B.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(3 points) Determine a linear equation over the variables x_1, x_2, x_3, x_4 whose solution set is $\text{Col } A$ (that is, (x_1, x_2, x_3, x_4) satisfies the linear equation if and only if the vector $[x_1 \ x_2 \ x_3 \ x_4]^T$ is in the column space of A).

Solution.

2 Linear Models

$$\{(-1, 1), (0, -1), (1, 1), (2, 3), (4, 5)\}$$

(3 points) Determine the equations for finding the best-fit curve of the form

$$\beta_0 + \beta_1 x^3 + \beta_2 (2^x)$$

(where β_0 , β_1 , and β_2 are parameters) for the above data using least-squares regression. That is, determine the design matrix X and vector of observations \mathbf{y} such that the least-squares solution of

$$X \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{y}$$

determines the parameters for best-fit curve.

Solution.

3 Exponentials of Matrices

In a homework problem we saw that if A is diagonalizable (i.e., it can be expressed as PDP^{-1} for a diagonal matrix D) then we can define $A^{1/2}$ as $PD^{1/2}P^{-1}$, where

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}^{1/2} = \begin{bmatrix} d_1^{1/2} & 0 & \dots & 0 & 0 \\ 0 & d_2^{1/2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1}^{1/2} & 0 \\ 0 & 0 & \dots & 0 & d_n^{1/2} \end{bmatrix}$$

That is, we can take the square root of each entry along the diagonal of D in the diagonalization of A . It turns out that for any function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any diagonalizable matrix A as above, we can define $f(A)$ as $Pf(D)P^{-1}$ where

$$f\left(\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}\right) = \begin{bmatrix} f(d_1) & 0 & \dots & 0 & 0 \\ 0 & f(d_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & f(d_{n-1}) & 0 \\ 0 & 0 & \dots & 0 & f(d_n) \end{bmatrix}$$

That is, we apply f to each entry along the diagonal of D in the diagonalization of A . We will use this fact to reason about exponentials of matrices. (*Problem continued on next page.*)¹

¹Credit to Vishesh Jain and Abhinit Sati for suggesting a version of this problem.

A.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(4 points) Determine a diagonalization of A . Your final solution should be in the form PDP^{-1} , where D is a diagonal matrix and P^{-1} is an explicit matrix, i.e., you must compute the inverse of P .

Solution.

(Problem 2A Continued)

B.

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

(2 points) Determine the matrix 2^A . Your final solution should be an explicit matrix, i.e, you must compute all matrix multiplications. (*Hint.* Use $f(x) = 2^x$.)

Solution.

C.

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -6 \\ 8 & -8 \end{bmatrix}$$

(4 points) Determine the characteristic polynomial of $2^A 2^B$. Give your solution in expanded form, i.e., not factored. (*Hint.* Don't try to compute this matrix product directly. Use properties of exponentiation).

Solution.

4 True/False Questions

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.²

- A. (1 point) For any matrix A with orthogonal columns, $A^T A = I$.
- ☐ True
☐ False
- B. (1 point) If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.
- ☐ True
☐ False
- C. (1 point) For any matrix A , if A is orthogonally diagonalizable, then so is A^T .
- ☐ True
☐ False
- D. (1 point) For any vector $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^n , $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$.
- ☐ True
☐ False
- E. (1 point) For any matrix A and vector \mathbf{v} , if \mathbf{v} is an eigenvector of the matrix A then it is also an eigenvector of the matrix A^2 .
- ☐ True
☐ False
- F. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{v} \in \mathbb{R}^n$, if $\|A\mathbf{v}\| = \mathbf{0}$ then $\mathbf{v} \in \text{Nul } A$.
- ☐ True
☐ False
- G. (1 point) For any matrix A , we have $\text{rank}(A) = \text{rank}(A^T)$.
- ☐ True
☐ False
- H. (1 point) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^n , if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- ☐ True
☐ False
- I. (1 point) For any square matrices A and B , we have $\det(AB^T) = \det(A)\det(B)$.
- ☐ True
☐ False
- J. (1 point) For any quadratic form $Q(\mathbf{x})$, the vector $\text{argmax}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$ is unique.
- ☐ True
☐ False

²Credit to Vishesh Jain and Abhinit Sati for suggesting some parts of this question.

K. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, then $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

☐ True

☐ False

L. (1 point) If A is a square matrix with strictly positive entries, then there is diagonal matrix D such that AD is stochastic.

☐ True

☐ False

5 Singular Value Decomposition

A.

$$A = \begin{bmatrix} 2 & -4 & 4 \\ 2 & 2 & 1 \\ -2 & 4 & -4 \end{bmatrix} \quad AA^T = \begin{bmatrix} 36 & 0 & -36 \\ 0 & 9 & 0 \\ -36 & 0 & 36 \end{bmatrix}$$

(4 points) Determine a *reduced* singular value decomposition of A , given that $6\sqrt{2}$ and 3 are its nonzero singular values and $\text{rank}(A) = 2$. (*Reminder.* If $U\Sigma V^T$ is an SVD of A and $\text{rank}(A) = r$, then we can get a reduced SVD of A by making Σ an $r \times r$ diagonal matrix and dropping columns of U and V so that the multiplication $U\Sigma V^T$ is well-defined.)

Solution.

(Problem 5A Continued)

B.

$$\mathbf{b} = \begin{bmatrix} 10 \\ 18 \\ -14 \end{bmatrix}$$

(3 points) Let A be the matrix from the previous part. Determine the length of the *shortest* least-squares solution to the equation $A\mathbf{x} = \mathbf{b}$. (*Hint.* Use the pseudoinverse of A , i.e., $A^+ = V\Sigma^{-1}U^T$, and keep in mind that matrices with orthonormal columns preserve lengths.)

Solution.

(Problem 5B Continued)

6 Dependence Relations

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$$

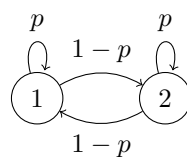
(5 points) Determine a dependence relation for the above vectors. That is, write $\mathbf{0}$ as a linear combination of the above vectors.

Solution.

(Problem 6 Continued)

7 Stochastic Matrices

A. Consider the following state diagram.



(2 points) Write the transition matrix T for the above diagram in terms of p . In the following parts, T will refer to this matrix.

Solution.

- B. (3 points) Given $0 < p < 1$, determine $\lim_{k \rightarrow \infty} T^k \mathbf{e}_1$ (where \mathbf{e}_1 is the first standard basis vector). Justify your answer.

Solution.

Justification.

- C. (4 points) Determine λ_2 , the *second* largest eigenvalue of T , and a corresponding eigenvector. (*Hint.* Keep in mind that T is symmetric and, hence, orthogonally diagonalizable.)³

Solution.

³The second largest eigenvalue tells us about the *rate of convergence* to the steady-state distribution. The smaller $|\lambda_2|$, the faster the convergence to the steady-state distribution.

(Problem 7C Continued)

- D. (4 points) Give a closed-form solution for $T^k \begin{bmatrix} 1-q \\ q \end{bmatrix}$ in terms of p , q and k (*Hint.* Again, keep in mind that A is symmetric and, hence, orthogonally diagonalizable).

Solution.

(Problem 7D Continued)

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