Midterm Exam (Version 1)

CAS CS 132: Geometric Algorithms

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- You will have approximately 75 minutes to complete this exam. Make sure to read every question, some are easier than others.
- Do not remove any pages from the exam.
- Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not recieve credit it you don't show your work.
- We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

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1 Linear Dependence

(6 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors such that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 5 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine a dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. That is, determine coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & -3 & 5 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & -1 & +2 & \end{array} \\ \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = 2x_3$

x_3 is free

$x_4 = 0$

Solution.

$$\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

2 True/False

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.

A. (1 point) For any matrices $A, B \in \mathbb{R}^{n \times n}$, if A and B are invertible, then so is $A + B$.

○ True

● False

B. (1 point) If a system ~~of linear equations~~ of linear equations has infinitely many solutions, then it must have more unknowns than equations.

○ True

● False

C. (1 point) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is the set $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_2, \mathbf{v}_3 + \mathbf{v}_2\}$.

● True

○ False

D. (1 point) If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.

● True

○ False

E. (1 point) For any matrix $A \in \mathbb{R}^{n \times n}$, if AA^T is symmetric, then so is A (recall that a matrix B is symmetric if $B = B^T$).

○ True

● False

F. (1 point) A system of linear equations with 4 variables and 3 equations cannot be inconsistent.

○ True

● False

G. (1 point) For any matrices $A, B \in \mathbb{R}^{n \times n}$, if AB is invertible, then so are A and B.

● True

○ False

H. (1 point) For any matrix $A \in \mathbb{R}^{n \times n}$, if the columns of A span $\mathbb{R}^n$, then the reduced echelon form of A has only 0s and 1s.

● True

○ False

I. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$, if $A = LU$ where L and U form an LU-factorization of A, then L is invertible.

● True

○ False

3 Linear Equations and Spans

(6 points) Determine a linear equation whose solution set is the span of the following two vectors. (*Note.* You may not use the cross product for this problem.)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & b_1 \\ 2^{-2} & 7^{-6} & b_2^{-2b_1} \\ 1^{-1} & 5^{-3} & b_3^{-b_1} \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 2^{-2} & b_3 - b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - b_1 - 2(b_2 - 2b_1) \end{bmatrix}$$

$-2(b_2 - b_1)$

$$x_3 - x_1 - 2x_2 + 4x_1 = 0$$

$$3x_1 - 2x_2 + x_3 = 0$$

Solution.

$$3x_1 - 2x_2 + x_3 = 0$$

4 Pseudoinverses

(6 points) Determine a matrix A and two values k and h such that

$$A\begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\qquad\text{and}\qquad\begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}A=\begin{bmatrix}1&0&0\\0&1&0\\h&k&0\end{bmatrix}$$

(*Hint.* First determine the rightmost column of A.)

APPROACH 1:

$A \in \mathbb{R}^{2\times 3}$ $\quad B=\begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}$ $\quad BA = B[\vec a_1\ \vec a_2\ \vec a_3] = [B\vec a_1\ B\vec a_2\ B\vec a_3] = \begin{bmatrix}1&0&0\\0&1&0\\h&k&0\end{bmatrix} \Rightarrow$

$B\vec a_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \vec a_3 = \begin{bmatrix}0\\0\end{bmatrix}$

$A=\begin{bmatrix}a&b&0\\c&d&0\end{bmatrix} \Rightarrow AB = \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&1\\2&3\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \Rightarrow$

$\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}1&1\\2&3\end{bmatrix}^{-1} = \dfrac{1}{3-2}\begin{bmatrix}3&-1\\-2&1\end{bmatrix} = \begin{bmatrix}3&-1\\-2&1\end{bmatrix}$

$A = \begin{bmatrix}3&-1&0\\-2&1&0\end{bmatrix}$, $\quad B\vec a_1 = B\begin{bmatrix}3\\-2\end{bmatrix} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix} + \begin{bmatrix}-2\\-6\\-4\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix} \Rightarrow h=5$

$B\vec a_2 = B\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\-2\\-3\end{bmatrix} + \begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix} \Rightarrow k=-1$

APPROACH 2:

$$\begin{bmatrix}1&1&1&0&0\\2&3&0&1&0\\3&2&h&k&0\end{bmatrix} \sim \begin{bmatrix}1&1&1&0&0\\0&1&-2&1&0\\0&-1&h-3&k&0\end{bmatrix} \sim \begin{bmatrix}1&1&1&0&0\\0&1&-2&1&0\\0&0&h-5&k+1&0\end{bmatrix} \sim \begin{bmatrix}1&0&3&-1&0\\0&1&-2&1&0\\0&0&h-5&k+1&0\end{bmatrix}$$

$\Rightarrow h=5,\ k=-1,\ a=3,\ b=-1,\ c=-2,\ d=1$

Solution.

$A = \begin{bmatrix}3&-1&0\\-2&1&0\end{bmatrix}$ $\qquad h = 5$ $\qquad k = 1$

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5 Linear Transformations

Consider the following linear transformations.

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_2 \\ x_1 - x_2 \end{bmatrix} \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ -x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

A. (3 points) Determine the matrix that implements $S \circ T$, the composition of S and T. That is, determine the matrix A such that the transformation $\mathbf{x} \mapsto S(T(\mathbf{x}))$ is equivalent to the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

APPROACH 1:

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \qquad S(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \vec{x}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix} \vec{x}$$

$$S \circ T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -4 & -6 \\ 2 & 3 & 4 \end{bmatrix} \vec{x}$$

APPROACH 2:

$$\underbrace{S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}_{= S(T(\vec{e}_1))} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \qquad \underbrace{S\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)}_{= S(T(\vec{e}_2))} = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \qquad \underbrace{S\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)}_{= S(T(\vec{e}_3))} = \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -4 & -6 \\ 2 & 3 & 4 \end{bmatrix}$$

B. (6 points) Determine a general form solution that describes the preimages of the following vector $\mathbf{b}$ under the linear transformation $S \circ T$. That is determine a general form solution where $S(T(\mathbf{x})) = \mathbf{b}$ if and only if $\mathbf{x}$ is in the solution set described by your general form solution.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 8 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \overset{+2}{-2} & \overset{+2}{-4} & \overset{+2}{-6} & \overset{+2}{8} \\ \overset{-2}{2} & \overset{-2}{3} & \overset{-2}{4} & \overset{-2}{-3} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 10 \\ 0 & \overset{-1}{1} & \overset{-2}{2} & \overset{+5}{-5} \end{bmatrix} \sim \begin{bmatrix} 1 & \overset{-1}{1} & \overset{-2}{1} & \overset{+5}{1} \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.

$x_1 = 6 + x_3$

$x_2 = -5 - 2x_3$

x_3 is free

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