Midterm Exam (Version 2)

CAS CS 132: Geometric Algorithms

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- ▷ You will have approximately 75 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Do not remove any pages from the exam.
- ▷ Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not recieve credit it you don't show your work.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

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1 Linear Equations and Spans

(6 points) Determine a linear equation whose solution set is the span of the following two vectors. (*Note.* You may not use the cross product for this problem.)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & b_1 \\ -1^{+1} & -2^{+3} & b_2^{+b_1} \\ 2^{-2} & 3^{-6} & b_3^{-2b_1} \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & b_2 + b_1 \\ 0 & -3^{+3} & b_3 - 2b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & b_1 \\ 0 & -1 & b_2 + b_1 \\ 0 & 0 & b_3 - 2b_1 + 3(b_2 + b_1) \end{bmatrix}$$

$+3(b_2+b_1)$

$$x_3 - 2x_1 + 3x_2 + 3x_1$$

$$x_1 + 3x_2 + x_3 = 0$$

Solution.

$$x_1 + 3x_2 + x_3 = 0$$

2 Pseudoinverses

(6 points) Determine a matrix A and two values k and h such that

$$A\begin{bmatrix}1 & -1\\ 3 & -2\\ -1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \text{and} \qquad \underbrace{\begin{bmatrix}1 & -1\\ 3 & -2\\ -1 & 3\end{bmatrix}}_{B} A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ h & k & 0\end{bmatrix}$$

(*Hint.* First determine the rightmost column of A.)

APPROACH 1:

$A \in \mathbb{R}^{2\times 3}$ $\quad BA = B[\vec{a}_1 \;\; \vec{a}_2 \;\; \vec{a}_3] = [B\vec{a}_1 \;\; B\vec{a}_2 \;\; B\vec{a}_3]$

$$B\vec{a}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \vec{a}_3 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$A = \begin{bmatrix}a & b & 0\\ c & d & 0\end{bmatrix} \Rightarrow AB = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & -1\\ 3 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & -1\\ 3 & -2\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 1\\ -3 & 1\end{bmatrix}$$

$$A = \begin{bmatrix}-2 & 1 & 0\\ -3 & 1 & 0\end{bmatrix} \qquad B\vec{a}_1 = \begin{bmatrix}1 & -1\\ 3 & -2\\ -1 & 3\end{bmatrix}\begin{bmatrix}-2\\ -3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ -7\end{bmatrix}$$

$$B\vec{a}_2 = \begin{bmatrix}1 & -1\\ 3 & -2\\ -1 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 2\end{bmatrix}$$

APPROACH 2:

$$\begin{bmatrix}1 & -1 & 1 & 0\\ 3^{-3} & -2^{+3} & 0^{-3} & 1\\ -1^{+1} & 3^{-1} & h^{+1} & k\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 1 & 0\\ 0 & 1 & -3 & 1\\ 0 & 2^{-2} & h+1^{+6} & k^{-2}\end{bmatrix} \sim \begin{bmatrix}1 & -1^{+1} & 1^{-3} & 0^{+1}\\ 0 & 1 & -3 & 1\\ 0 & 0 & h+7 & k-2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2 & 1\\ 0 & 1 & -3 & 1\\ 0 & 0 & h+7 & k=2\end{bmatrix}$$

$\Rightarrow h = -7,\; k = 2,\; a = -2,\; b = 1,\; c = -3,\; d = 1$

Solution.

$$A = \begin{bmatrix}-2 & 1 & 0\\ -3 & 1 & 0\end{bmatrix} \qquad h = -7 \qquad k = 2$$

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3 Linear Transformations

Consider the following linear transformations.

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ -x_2 \\ x_1 + x_2 \end{bmatrix} \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ -x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

A. (3 points) Determine the matrix that implements $S \circ T$, the composition of S and T. That is, determine the matrix A such that the transformation $\mathbf{x} \mapsto S(T(\mathbf{x}))$ is equivalent to the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

APPROACH 1:

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad S(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix} \vec{x}$$

$$(S \circ T)(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \vec{x}$$

APPROACH 2:

$$\underset{\displaystyle S(T(\vec{e}_1))}{\underset{\shortparallel}{S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \underset{\displaystyle S(T(\vec{e}_2))}{\underset{\shortparallel}{S\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)}} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad \underset{\displaystyle S(T(\vec{e}_3))}{\underset{\shortparallel}{S\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)}} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

B. (6 points) Determine a general form solution that describes the preimages of the following vector $\mathbf{b}$ under the linear transformation $S \circ T$. That is determine a general form solution where $S(T(\mathbf{x})) = \mathbf{b}$ if and only if $\mathbf{x}$ is in the solution set described by your general form solution.

$$\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & 4 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1^{-1} & 2^{-1} & 3^{-1} & 1^{-2} \\ 0 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \overset{-1}{1} & \overset{-2}{1} & \overset{+1}{2} \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.

$x_1 = 3 + x_3$

$x_2 = -1 - 2x_3$

x_3 is free

4 True/False

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.

A. (1 point) A system of linear equations with 3 variables and 4 equations cannot be consistent.

○ True

● False

B. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$, if $A = LU$ where L and U form an LU-factorization of A, and if L is invertible, then so is A.

○ True

● False

C. (1 point) For any matrices $A, B \in \mathbb{R}^{n\times n}$, if A and B are invertible, then so is AB.

● True

○ False

D. (1 point) If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.

● True

○ False

E. (1 point) If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then it has infinitely many solutions.

● True

○ False

F. (1 point) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3\}$.

○ True

● False

G. (1 point) For any matrix $A \in \mathbb{R}^{m\times n}$, the matrix AA^T is symmetric (recall that a matrix B is symmetric if $B = B^T$).

● True

○ False

H. (1 point) For any matrix $A \in \mathbb{R}^{m\times n}$, if $m < n$, and the columns of A span $\mathbb{R}^m$, then the reduced echelon form of A has only 0s and 1s.

○ True

● False

I. (1 point) For any matrices $A, B \in \mathbb{R}^{n\times n}$, if $AB = I$, then $BA = I$.

● True

○ False

5 Linear Dependence

(6 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors such that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 3 & 8 \\ 0 & -2 & -2 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine a dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. That is, determine coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{0}$.

$$\begin{bmatrix} 2 & -3 & 3 & 8 \\ 0 & -2 & -2 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & +3 & +3 & \end{array} \\ \begin{bmatrix} 2 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -3x_3$

$x_2 = -x_3$

x_3 is free

$x_4 = 0$

Solution.

$$-3\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$$

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