Midterm Exam 2 (Version 1)

CAS CS 132: Geometric Algorithms

Name:

BUID:

- ▷ You will have approximately 75 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Do not remove any pages from the exam.
- ▷ Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not receive credit if you don't show your work.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

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1 Matrix Inverses

$$A = \begin{bmatrix} -2 & -2 & 7 \\ 1 & 2 & -1 \\ -2 & -3 & 5 \end{bmatrix}$$

A. (7 points) Determine the inverse of A, i.e., determine the matrix B such that $AB = BA = I$.

Solution.

B. (3 points) Use the matrix you determined in Part A to calculate the solution to the matrix equation $A\mathbf{x} = [1\ 1\ 1]^T$. For partial credit, determine the solution to the matrix equation $A\mathbf{x} = [1\ 1\ 1]^T$ using Gaussian elimination.

Solution.

2 True/False

(10 points) Determine if each of the following statements is **True** or **False**. Bubble in your answers below.

A. (1 point) For any stochastic matrix $A \in \mathbb{R}^{n \times n}$, if A has a unique steady-state distribution, then A is regular.

- ◯ True
- ◯ False

B. (1 point) If $\operatorname{rank} A = 1$, then the matrix equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

- ◯ True
- ◯ False

C. (1 point) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

- ◯ True
- ◯ False

D. (1 point) For matrices A, B, C in $\mathbb{R}^{n \times n}$, if $AB = AC$ then $B = C$.

- ◯ True
- ◯ False

E. (1 point) For a subspace H of $\mathbb{R}^n$ and vector $\mathbf{v} \in \mathbb{R}^n$, if $\mathbf{v} \notin H$, then $\dim H < n$.

- ◯ True
- ◯ False

F. (1 point) For a matrix $A \in \mathbb{R}^{2026 \times 2027}$, it is possible for the rank of A to be the same as to $\dim(\operatorname{Nul} A)$.

- ◯ True
- ◯ False

G. (1 point) If A is invertible, then both the L-part and the U-part of an LU-factorization of A are invertible.

- ◯ True
- ◯ False

H. (1 point) For any stochastic matrix $A \in \mathbb{R}^{n \times n}$, if A is regular then $\dim(\operatorname{Nul}(A - I)) = 1$.

- ◯ True
- ◯ False

I. (1 point) For matrices A and B in $\mathbb{R}^{m \times n}$, let $\begin{bmatrix} A \\ B \end{bmatrix}$ denote the $\mathbb{R}^{2m \times n}$ obtained by stacking A and B vertically. For any vector $\mathbf{b} \in \mathbb{R}^n$, if $A\mathbf{x} = \mathbf{b}$ is consistent and $B\mathbf{x} = \mathbf{b}$ is consistent then so is $\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix}$.

- ◯ True
- ◯ False

J. (1 point) Given linear transformations $S : \mathbb{R}^m \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$, if $m < n$, then it is possible for $T \circ S$ to be invertible (recall that $(T \circ S)(\mathbf{v}) = T(S(\mathbf{v}))$).

- ◯ True
- ◯ False

3 Subspaces

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x + 5y \\ -x - 2y - z \\ x + 3y - z \end{bmatrix}$$

A. (2 points) Determine the matrix A that implements the above transformation, i.e., determine the matrix A such that the above transformation is equivalent to $\mathbf{x} \mapsto A\mathbf{x}$.

Solution.

B. (4 points) Determine a basis for Nul A.

Solution.

You may also use this page to show your work for both Part B and Part C if necessary.

C. (2 points) Determine a basis for $\operatorname{Col} A$.

Solution.

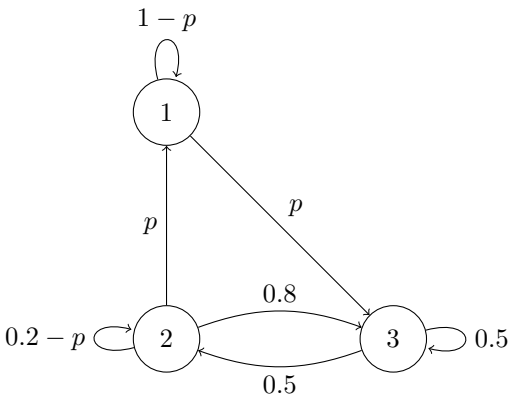
D. (2 points) Let T denote the transformation from Part A and let H be the subspace

$$\left\{\mathbf{v} : \mathbf{v} \in \operatorname{ran} T \text{ and } \mathbf{e}_3^T \mathbf{v} = 0\right\}$$

That is, H is **the subspace of vectors in the range of T whose third components are 0.** Note that H is a subspace of $\operatorname{ran} T$, which is a subspace of $\mathbb{R}^3$. Determine a basis for H.

Solution.

4 Markov Chains



A. (2 points) Determine the transition matrix A of the above state diagram in terms of p. The transitions out of state 1 should appear in the first column of A, out of 2 in the second column, and out of 3 in the third column.

Solution.

B. (2 points) Is the matrix A regular in the case that $p = 0$? Justify your answer.

Solution. ◯ Yes ◯ No

Justification.

C. (4 points) Determine $\lim_{k\to\infty} A^k[0.2\ 0.4\ 0.4]^T$ in the case that $p = 0.1$.

Solution.

D. (2 points) Determine $\lim_{k\to\infty} A^k[0.2\ 0.4\ 0.4]^T$ in the case that $p = 0$. For partial credit, determine a general form solution that describes the steady-state vectors of A in the case that $p = 0$ (the steady state vectors of a matrix A are the vectors $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$).

Solution.

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