Final Exam

CAS CS 132: Geometric Algorithms

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- You will have 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- Do not remove any pages from the exam.
- Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not receive credit if you don't show your work.
- We will not look at any work on the pages marked "*This page is intentionally left blank.*" You can use these pages for scratch work.

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1 Linear Systems

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 0 & -4 & -1 \\ 2 & 1 & 6 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -6 \\ -7 \\ -7 \end{bmatrix}$$

A. (5 points) Determine a general form solution for the equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 1 & 2 & -1 & -6 \\ -1\,{}^{+1} & 0\,{}^{+1} & -4\,{}^{+2} & -1\,{}^{-1} & -7\,{}^{-6} \\ 2\,{}^{-2} & 1\,{}^{-2} & 6\,{}^{-4} & 1\,{}^{+2} & -7\,{}^{+12} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -1 & -6 \\ 0 & 1 & -2 & -2 & -13 \\ 0 & -1\,{}^{+1} & 2\,{}^{-2} & 3\,{}^{-2} & 5\,{}^{-13} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -1\,{}^{+1} & -6\,{}^{-8} \\ 0 & 1 & -2 & -2\,{}^{+2} & -13\,{}^{-16} \\ 0 & 0 & 0 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1\,{}^{-1} & 2\,{}^{+2} & 0 & -14\,{}^{+29} \\ 0 & 1 & -2 & 0 & -29 \\ 0 & 0 & 0 & 1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 15 \\ 0 & 1 & -2 & 0 & -29 \\ 0 & 0 & 0 & 1 & -8 \end{bmatrix}$$

$x_1 = 15 - 4x_3$

$x_2 = -29 + 2x_3$

x_3 is free

$x_4 = -8$

B. *The matrix A is reproduced for convenience.*

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 0 & -4 & -1 \\ 2 & 1 & 6 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

(5 points) Determine if the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent. If it is, write a dependence relation for this set of vectors, i.e., express $\mathbf{0}$ as a nontrivial linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. If it is not, then justify your answer.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$4\vec{x}_1 - 2x_2 - x_3 = \vec{0}$$

2 True/False

(10 points) Determine if each of the following statements is **True** or **False**. Bubble in your answers below.

A. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$, if A has an eigenbasis of $\mathbb{R}^n$ then A is symmetric.

○ True

● False

B. (1 point) If $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto, then A is a square matrix.

● True

○ False

C. (1 point) For any vector $\mathbf{u} \in \mathbb{R}^n$, if $\|\mathbf{u}\| = 1$ then the matrix that implements orthogonal projection onto $\mathbf{u}$ is orthogonally diagonalizable.

● True

○ False

D. (1 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if $m < n$ then the columns of A are linearly dependent.

● True

○ False

E. (1 point) For any matrix $U \in \mathbb{R}^{n\times n}$, if U is orthogonal (i.e., U has orthonormal columns) then $\det U = \pm 1$.

● True

○ False

F. (1 point) If matrices A and B in $R^{n\times n}$ are singular, then $\det A = \det B$.

● True

○ False

G. (1 points) For any matrix $A \in \mathbb{R}^{n\times n}$, if A is regular and stochastic then $\dim \mathrm{Nul}(A - I) = 1$.

● True

○ False

H. (1 point) Every positive-definite symmetric matrix is invertible.

● True

○ False

I. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$ with positive entries, there is a matrix B such that AB is stochatic.

● True

○ False

J. (1 point) If the augmented matrix of a linear system has a pivot position in every column, then the linear system is inconsistent.

● True

○ False

3 Short Answer

A. (3 points) Let R denote the 2×2 matrix which rotates a points *clockwise* about the origin by $\frac{\pi}{7}$ radians, and let A denote the matrix which reflects points across the line $y = x$. Determine the matrix B such that $RA = AB$. Your solution should be an explicit 2×2 matrix. You may leave any sines and cosines unevaluated.

$$R = \begin{bmatrix} \cos\frac{\pi}{7} & \sin\frac{\pi}{7} \\ -\sin\frac{\pi}{7} & \cos\frac{\pi}{7} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-1}$$

$$B = A^{-1}RA = ARA$$

$$B = \boxed{\begin{bmatrix} \cos\frac{\pi}{7} & -\sin\frac{\pi}{7} \\ \sin\frac{\pi}{7} & \cos\frac{\pi}{7} \end{bmatrix}}$$

B.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(3 points) Determine the largest eigenvalue of A and a corresponding eigenvector. *Hint.* Recall that the solutions to a quadratic equation of the form $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\det(A - \lambda I) = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$A - \lambda_1 I = \begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \sim \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{-1+5}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix} \sim \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{2}{1-\sqrt{5}} \\ 0 & 0 \end{bmatrix}$$

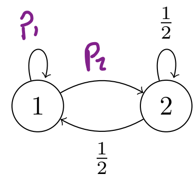
$$= \begin{bmatrix} 1 & \frac{2+2\sqrt{5}}{1-5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\left(\frac{1+\sqrt{5}}{2}\right) \\ 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1+\sqrt{5}}{2} x_2$

x_2 is free

$$\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

C. (4 points) Consider the following transition diagram.



Determine the probability of transitioning from state 1 to state 2, given that the stationary distribution of this transition system is $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$.

$$\begin{bmatrix} p_1 & 1/2 \\ p_2 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

$$3/4\, p_1 + \frac{1}{8} = 3/4$$

$$3 p_1 + \frac{1}{2} = 3$$

$$3 p_1 = \frac{5}{2}$$

$$p_1 = \frac{5}{6} \qquad p_2 = \frac{1}{6}$$

D.

$$A = \begin{bmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

(4 points) Determine a diagonalization of A. Your solution must be given in the form of three explicit matrices P, D and P^{-1} such that D is a diagonal matrix and $A = PDP^{-1}$. In particular, P^{-1} must be given explicitly for full credit. You can receive partial credit if you calculate only P and D. *Hint.* One of the eigenvalues of A is 6 and one of the eigenvectors of A is $[1\ 1\ 1]^T$. Also note that A is symmetric.

$$A - 6I = \begin{bmatrix} -1 & 5 & 2 \\ \overset{-5}{5} & \overset{+25}{-1} & \overset{+10}{2} \\ \overset{-2}{2} & \overset{+10}{2} & \overset{+4}{2} \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 2 \\ 0 & 24 & 12 \\ 0 & 12 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & \overset{+5}{-5} & \overset{+5/2}{-2} \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -1/2\, x_3$
$x_2 = -1/2\, x_3$
x_3 is free

$$v = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} \qquad \lambda_1 = 12$$

$$A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \vec{0} \quad \text{(by inspection)}$$

$$A = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}}_{P^{-1} = P^T}$$

4 Singular Value Decomposition

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 1 \end{bmatrix}$$

(8 points) Determine a singular value decomposition of A. *Hint.* Think in terms of A^T.

$AA^T = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix}$ $\quad \det(A^TA - \lambda I) = (\lambda-10)^2 - 64 = \lambda^2 - 20\lambda + 100 - 64$

$= \lambda^2 - 20\lambda + 36$

$= (\lambda - 18)(\lambda - 2)$

$AA^T - 18\lambda = \begin{bmatrix} -8 & -8 \\ -8 & -8 \end{bmatrix}$ $\quad x_1 = -x_2$, x_2 is free $\quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$AA^T - 2I = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$ $\quad x_1 = x_2$, x_2 is free $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$A^T = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{V^T}$$

$$A = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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