Final Exam

CAS CS 132: Geometric Algorithms

Name:

BUID:

- ▷ You will have 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Do not remove any pages from the exam.
- ▷ Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not receive credit if you don't show your work.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You can use these pages for scratch work.

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1 Linear Systems

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 0 & -4 & -1 \\ 2 & 1 & 6 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -6 \\ -7 \\ -7 \end{bmatrix}$$

A. (5 points) Determine a general form solution for the equation $A\mathbf{x} = \mathbf{b}$.

B. *The matrix A is reproduced for convenience.*

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 0 & -4 & -1 \\ 2 & 1 & 6 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

(5 points) Determine if the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent. If it is, write a dependence relation for this set of vectors, i.e., express $\mathbf{0}$ as a nontrivial linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. If it is not, then justify your answer.

2 True/False

(10 points) Determine if each of the following statements is **True** or **False**. Bubble in your answers below.

A. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$, if A has an eigenbasis of $\mathbb{R}^n$ then A is symmetric.

◯ True
◯ False

B. (1 point) If $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto, then A is a square matrix.

◯ True
◯ False

C. (1 point) For any vector $\mathbf{u} \in \mathbb{R}^n$, if $\|\mathbf{u}\| = 1$ then the matrix that implements orthogonal projection onto $\mathbf{u}$ is orthogonally diagonalizable.

◯ True
◯ False

D. (1 points) For any matrix $A \in \mathbb{R}^{m\times n}$, if $m < n$ then the columns of A are linearly dependent.

◯ True
◯ False

E. (1 point) For any matrix $U \in \mathbb{R}^{n\times n}$, if U is orthogonal (i.e., U has orthonormal columns) then $\det U = \pm 1$.

◯ True
◯ False

F. (1 point) If matrices A and B in $R^{n\times n}$ are singular, then $\det A = \det B$.

◯ True
◯ False

G. (1 points) For any matrix $A \in \mathbb{R}^{n\times n}$, if A is regular and stochastic then $\dim \mathrm{Nul}(A - I) = 1$.

◯ True
◯ False

H. (1 point) Every positive-definite symmetric matrix is invertible.

◯ True
◯ False

I. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$ with positive entries, there is a matrix B such that AB is stochatic.

◯ True
◯ False

J. (1 point) If the augmented matrix of a linear system has a pivot position in every column, then the linear system is inconsistent.

◯ True
◯ False

3 Short Answer

A. (3 points) Let R denote the 2×2 matrix which rotates a points *clockwise* about the origin by $\frac{\pi}{7}$ radians, and let A denote the matrix which reflects points across the line $y = x$. Determine the matrix B such that $RA = AB$. Your solution should be an explicit 2×2 matrix. You may leave any sines and cosines unevaluated.

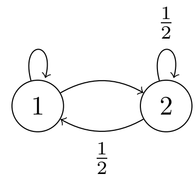
B.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(3 points) Determine the largest eigenvalue of A and a corresponding eigenvector. *Hint.* Recall that the solutions to a quadratic equation of the form $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

C. (4 points) Consider the following transition diagram.



Determine the probability of transitioning from state 1 to state 2, given that the stationary distribution of this transition system is $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$.

D.

$$A = \begin{bmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

(4 points) Determine a diagonalization of A. Your solution must be given in the form of three explicit matrices P, D and P^{-1} such that D is a diagonal matrix and $A = PDP^{-1}$. In particular, P^{-1} must be given explicitly for full credit. You can receive partial credit if you calculate only P and D. *Hint.* One of the eigenvalues of A is 6 and one of the eigenvectors of A is $[1\ 1\ 1]^T$. Also note that A is symmetric.

4 Singular Value Decomposition

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 1 \end{bmatrix}$$

(8 points) Determine a singular value decomposition of A. *Hint.* Think in terms of A^T.

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