Final Exam

CAS CS 132: Geometric Algorithms

Name: Nathan Mull

BUID: 12345678

- ▷ You will have 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Do not remove any pages from the exam.
- ▷ Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not receive credit if you don't show your work.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You can use these pages for scratch work.

Problem	Points
1	12
2	10
3	14
4	14
5	10
Total	60

This page is intentionally left blank.

1 Linear Systems

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ -1 & 0 & -3 & 2 \\ 2 & 3 & 12 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 11 \end{bmatrix}$$

A. (8 points) Determine if $\mathbf{b} \in \operatorname{Col} A$, i.e., if the vector $\mathbf{b}$ is in the span of the columns of A. If it is, then express $\mathbf{b}$ as a linear combination of the columns of A (you should use $\mathbf{a}_i$ to denotes the i^{th} column of A). If it is not, then justify your answer.

$$\begin{bmatrix} 1 & 1 & 5 & 1 & 5 \\ -1^{+1} & 0^{+1} & -3^{+5} & 2^{+1} & -3^{+5} \\ \overset{-2}{2} & \overset{-2}{3} & \overset{-10}{12} & \overset{-2}{4} & \overset{-10}{11} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 1^{-1} & 2^{-2} & 2^{-3} & 1^{-2} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & \overset{-1}{1} & \overset{-1}{5} \\ 0 & 1 & 2 & \overset{-3}{3} & \overset{-3}{2} \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \overset{-1}{1} & \overset{-2}{5} & 0 & \overset{+1}{4} \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 5 - 3x_3$

$x_2 = -1 - 2x_3$

x_3 is free

$x_4 = 1$

Solution.

$5\vec{a}_1 - \vec{a}_2 + \vec{a}_4$

B. *The matrix A is reproduced for convenience.*

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ -1 & 0 & -3 & 2 \\ 2 & 3 & 12 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

(4 points) Determine if the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent. If it is, write a dependence relation for this set of vectors, i.e., express $\mathbf{0}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$. If it is not, then justify your answer.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by part A}$$

Solution.

Linearly independent. $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$ is a basis of $\operatorname{Col}(A)$.

2 True/False

(10 points) Determine if each of the following statements is **True** or **False**. Bubble in your answers below. For problems G through J on the following page, you can provide justification or a counterexample for 1 point of extra credit. **Important:** The extra credit is *only* for the justification/counterexample. The true/false question on its own is still part of the regular exam.

A. (1 points) For any matrix $A \in \mathbb{R}^{m \times n}$, if $m > n$ then the columns of A are linearly dependent.

- ○ True
- ● False

B. (1 point) Let $E \in \mathbb{R}^{n \times n}$ denote the elementary matrix that implements $R_1 \leftarrow 3R_1$. For any matrix $A \in \mathbb{R}^{n \times n}$, if $\det(EA) = 1$ then $\det A^{-1} = \frac{1}{3}$.

- ○ True
- ● False

C. (1 point) For any matrix $U \in \mathbb{R}^{n \times n}$, if U is orthogonal (i.e., U has orthonormal columns) then $\det U = \pm 1$.

- ● True
- ○ False

D. (1 point) For any matrix $A \in \mathbb{R}^{n \times n}$, if A has an eigenbasis of $\mathbb{R}^n$ then A is symmetric.

- ○ True
- ● False

E. (1 point) A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if A^T is invertible.

- ● True
- ○ False

F. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$, if A is symmetric then an orthogonal diagonalization of A is also a singular value decomposition of A.

- ● True
- ○ False

G. (1 points) For any matrix $A \in \mathbb{R}^{n\times n}$, if A is regular and stochastic then $\dim \operatorname{Nul}(A - I) = 1$.

- ● True
- ○ False

Extra Credit. (1 point) Provide justification or a counterexample.

Regular stochastic matrices have unique steady states.

H. (1 point) Every positive-definite symmetric matrix is invertible.

- ● True
- ○ False

Extra Credit. (1 point) Provide justification or a counterexample.

0 is not an eigenvalue $\Leftrightarrow$ A is invertible

I. (1 point) For any matrix $A \in \mathbb{R}^{n\times n}$ and diagonal matrix $D \in \mathbb{R}^{n\times n}$, it must be that $AD = DA$.

- ○ True
- ● False

Extra Credit. (1 point) Provide justification or a counterexample.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

J. (1 point) For any vector $\mathbf{u} \in \mathbb{R}^n$, if $\|\mathbf{u}\| = 1$ then the matrix $\mathbf{u}\mathbf{u}^T$ implements projection onto span$\{\mathbf{u}\}$.

- ● True
- ○ False

Extra Credit. (1 point) Provide justification or a counterexample.

$$\operatorname{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{u}, \vec{v} \rangle \vec{u}}{\|\vec{u}\|^2} = (\vec{u}^T \vec{v})\vec{u} = (\vec{u}^T \vec{u})\vec{v}$$

3 Rank-Nullity

$$A = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 2 & 10 & 6 & -4 & 1 \\ -7 & -3 & -5 & -6 & 3 \end{bmatrix}$$

A. (8 points) Determine the characteristic polynomial of A. Your solution must be given in fully factored form.

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 0 & 0 & 0 & 0 \\ * & 2-\lambda & 0 & 0 & 0 \\ * & * & -1-\lambda & 0 & 0 \\ * & * & * & -4-\lambda & 1 \\ * & * & * & -6 & 3-\lambda \end{bmatrix} \xrightarrow{R_4 \leftarrow (3-\lambda)R_4} \begin{bmatrix} -3-\lambda & 0 & 0 & 0 & 0 \\ * & 2-\lambda & 0 & 0 & 0 \\ * & * & -1-\lambda & 0 & 0 \\ * & * & * & (-4-\lambda)(3-\lambda) & (3-\lambda) \\ * & * & * & -6 & (3-\lambda) \end{bmatrix}$$

$$\begin{bmatrix} -3-\lambda & 0 & 0 & 0 & 0 \\ * & 2-\lambda & 0 & 0 & 0 \\ * & * & -1-\lambda & 0 & 0 \\ * & * & * & (-4-\lambda)(3-\lambda)+6 & 0 \\ * & * & * & -6 & (3-\lambda) \end{bmatrix}$$

$$\det(A-\lambda I) = \frac{-1}{\cancel{(3-\lambda)}}(\lambda+3)(\lambda-2)(\lambda+1)\cancel{(3-\lambda)}(\lambda^2+\lambda-12+6)$$

$= \,''$ $\quad '' \,(\lambda+3)(\lambda-2)$

Solution.

$$-(\lambda+3)^2(\lambda-2)^2(\lambda+1)$$

B. (2 points) Determine the rank of A (i.e., the dimension of Col A) and the nullity of A (i.e, the dimension of Nul A). Justify your answer with a single English sentence.

Solution.

$\operatorname{rank} A = 5$ $\qquad\qquad$ $\dim \operatorname{Nul} A = 0$

Justification. 0 is not an eigenvalue of A.

C. (2 points) Determine the rank of $A + I$. Not. error during exam

$$\operatorname{rank}(A+I) + \underbrace{\dim(\operatorname{Nul}(A+I))}_{=1} = 5$$

$$\operatorname{rank}(A-I) = 5$$

Solution.

$\operatorname{rank}(A + I) = 4$

$$B = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 2 & 10 & 6 & -4 & 1 \\ -7 & -3 & -5 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D. (2 points) Let B be the matrix formed by appending a row of zeros to the bottom of A. Determine $\operatorname{rank} B$, $\dim \operatorname{Nul} B$ and $\dim \operatorname{Nul} B^T$.

rank (B) = rank (A) (same # of pivots)

$\Downarrow$

dim Nul (B) = dim Nul (A)

$$\underbrace{\operatorname{rank}(B^T)}_{= \operatorname{rank}(B) = 5} + \underbrace{\dim \operatorname{Nul}(B^T)}_{= \boxed{1}} = 6$$

Solution.

rank B = 5 $\quad$ dim Nul B = 0 $\quad$ dim Nul B^T = 1

This page is intentionally left blank.

4 Singular Value Decomposition

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{bmatrix} \qquad V = \begin{bmatrix} \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{6}/3 \end{bmatrix}$$

A. (8 points) Determine left singular vectors of A **associated with nonzero singular values**, i.e., determine the columns of U in the *reduced* SVD of A. *Hint.* The right singular vectors of A associated with nonzero singular values of A are the columns of V, i.e., V is part of the reduced SVD of A.

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|} \qquad A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} \qquad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|} \qquad A\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

B. (6 points) Determine the singular values of A. *Hint.* It's easier to do this *without* determining $A^T A$ or AA^T.

$$\sigma_1 = \|A\vec{v}_1\| = \left\| \begin{bmatrix} 2\sqrt{3} \\ 0 \\ 2\sqrt{3} \end{bmatrix} \right\| = \sqrt{12+12} = 2\sqrt{6}$$

$$\sigma_2 = \|A\vec{v}_2\| = \left\| \begin{bmatrix} 0 \\ \sqrt{6} \\ 0 \end{bmatrix} \right\| \quad \sqrt{6}$$

$$\sigma_3 = 0$$

Solution.

$2\sqrt{6}$, $\sqrt{6}$, 0

C. **Extra Credit.** (2 points) Determine the matrix that implements the orthogonal projection onto Col A.

$$UU^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \sqrt{2}/2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

5 Stationary Distributions

(10 points) Let A be a matrix such that the following equalities holds.

$$A\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}6\\2\end{bmatrix} \qquad A^2\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$

Determine $\lim_{k\to\infty} A^k \begin{bmatrix}0.1\\0.9\end{bmatrix}$. *Hint.* First determine the matrix A.

$$A\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} \qquad A\begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$

$$A\begin{bmatrix}4\\0\end{bmatrix} = A\begin{bmatrix}6\\2\end{bmatrix} - A\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0.5\\0.5\end{bmatrix}$$

$$A\begin{bmatrix}0\\2\end{bmatrix} = A\begin{bmatrix}2\\2\end{bmatrix} - A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$A = \begin{bmatrix}0.5 & 1\\0.5 & 0\end{bmatrix}$$

$$A - I = \begin{bmatrix}-0.5 & 1\\0.5 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -2\\0 & 0\end{bmatrix}$$

$x_1 = 2x_2$

x_2 is free

$$x_1 + x_2 = 1 \Rightarrow 2x_2 + x_2 = 1 \Rightarrow x_2 = 1/3, \quad x_1 = 2/3$$

Problem 5 continued. The input-output behavior of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ *is reproduced for convenience.*

$$A \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad A^2 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

This page is intentionally left blank.