Final Exam

CAS CS 132: Geometric Algorithms

Name:

BUID:

- ▷ You will have 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Do not remove any pages from the exam.
- ▷ Your final solution must appear in the solution boxes for each problem. **Only include your final solution in the solution box. You must show your work outside of the solution box.** You will not receive credit if you don't show your work.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You can use these pages for scratch work.

Problem	Points
1	12
2	10
3	14
4	14
5	10
Total	60

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1 Linear Systems

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ -1 & 0 & -3 & 2 \\ 2 & 3 & 12 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 11 \end{bmatrix}$$

A. (8 points) Determine if $\mathbf{b} \in \operatorname{Col} A$, i.e., if the vector $\mathbf{b}$ is in the span of the columns of A. If it is, then express $\mathbf{b}$ as a nontrivial linear combination of the columns of A (you should use $\mathbf{a}_i$ to denotes the i^{th} column of A). If it is not, then justify your answer.

Solution.

B. *The matrix A is reproduced for convenience.*

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ -1 & 0 & -3 & 2 \\ 2 & 3 & 12 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

(4 points) Determine if the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent. If it is, write a dependence relation for this set of vectors, i.e., express $\mathbf{0}$ as a nontrivial linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$. If it is not, then justify your answer.

Solution.

2 True/False

(10 points) Determine if each of the following statements is **True** or **False**. Bubble in your answers below. For problems G through J on the following page, you can provide justification or a counterexample for 1 point of extra credit. **Important:** The extra credit is *only* for the justification/counterexample. The true/false question on its own is still part of the regular exam.

A. (1 points) For any matrix $A \in \mathbb{R}^{m \times n}$, if $m > n$ then the columns of A are linearly dependent.

◯ True

◯ False

B. (1 point) Let $E \in \mathbb{R}^{n \times n}$ denote the elementary matrix that implements $R_1 \leftarrow 3R_1$. For any matrix $A \in \mathbb{R}^{n \times n}$, if $\det(EA) = 1$ then $\det A^{-1} = \frac{1}{3}$.

◯ True

◯ False

C. (1 point) For any matrix $U \in \mathbb{R}^{n \times n}$, if U is orthogonal (i.e., U has orthonormal columns) then $\det U = \pm 1$.

◯ True

◯ False

D. (1 point) For any matrix $A \in \mathbb{R}^{n \times n}$, if A has an eigenbasis of $\mathbb{R}^n$ then A is symmetric.

◯ True

◯ False

E. (1 point) A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if A^T is invertible.

◯ True

◯ False

F. (1 point) For any matrix $A \in \mathbb{R}^{m \times n}$, if A is symmetric then an orthogonal diagonalization of A is also a singular value decomposition of A.

◯ True

◯ False

G. (1 points) For any matrix $A \in \mathbb{R}^{n \times n}$, if A is regular and stochastic then $\dim \mathrm{Nul}(A - I) = 1$.

◯ True

◯ False

Extra Credit. (1 point) Provide justification or a counterexample.

H. (1 point) Every positive-definite symmetric matrix is invertible.

◯ True

◯ False

Extra Credit. (1 point) Provide justification or a counterexample.

I. (1 point) For any matrix $A \in \mathbb{R}^{n \times n}$ and diagonal matrix $D \in \mathbb{R}^{n \times n}$, it must be that $AD = DA$.

◯ True

◯ False

Extra Credit. (1 point) Provide justification or a counterexample.

J. (1 point) For any vector $\mathbf{u} \in \mathbb{R}^n$, if $\|\mathbf{u}\| = 1$ then the matrix $\mathbf{u}\mathbf{u}^T$ implements projection onto $\mathrm{span}\{\mathbf{u}\}$.

◯ True

◯ False

Extra Credit. (1 point) Provide justification or a counterexample.

3 Rank-Nullity

$$A = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 2 & 10 & 6 & -4 & 1 \\ -7 & -3 & -5 & -6 & 3 \end{bmatrix}$$

A. (8 points) Determine the characteristic polynomial of A. Your solution must be given in fully factored form.

Solution.

B. (2 points) Determine the rank of A (i.e., the dimension of $\operatorname{Col} A$) and the nullity of A (i.e, the dimension of $\operatorname{Nul} A$). Justify your answer with a single English sentence.

Solution.

$\operatorname{rank} A =$ $\qquad\qquad$ $\dim \operatorname{Nul} A =$

Justification.

C. (2 points) Determine the rank of $A - I$.

Solution.

$\operatorname{rank}(A - I) =$

$$B = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 2 & 10 & 6 & -4 & 1 \\ -7 & -3 & -5 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D. (2 points) Let B be the matrix formed by appending a row of zeros to the bottom of A. Determine $\operatorname{rank} B$, $\dim \operatorname{Nul} B$ and $\dim \operatorname{Nul} B^T$.

Solution.

$\operatorname{rank} B =$ $\dim \operatorname{Nul} B =$ $\dim \operatorname{Nul} B^T =$

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4 Singular Value Decomposition

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{bmatrix} \qquad V = \begin{bmatrix} \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{6}/3 \end{bmatrix}$$

A. (8 points) Determine left singular vectors of A **associated with nonzero singular values**, i.e., determine the columns of U in the *reduced* SVD of A, where $U\Sigma V^T$ is the reduced SVD of A. Recall that the right singular vectors of A associated with nonzero singular values of A are the columns of V.

Solution.

B. (6 points) Determine the singular values of A. *Hint.* It's easier to do this *without* determining $A^T A$ or AA^T.

Solution.

C. **Extra Credit.** (2 points) Determine the matrix that implements orthogonal projection onto Col A.

Solution.

5 Stationary Distributions

(10 points) Let A be a matrix such that the following equalities holds.

$$A \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad A^2 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Determine $\lim_{k \to \infty} A^k \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$. *Hint.* First determine the matrix A.

Problem 5 continued. The input-output behavior of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ *is reproduced for convenience.*

$$A \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad A^2 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution.

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