Problems and Exercises

`v0.0.2`

`CAS CS 132`: *Geometric Algorithms*

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Contents

Preface

The following is a collection of linear algebraic problems and exercises for the course `CAS CS 132`: *Geometric Algorithms* at Boston University. As a student of this course, you will hopefully work through nearly every problem in this document. The problems are organized by topic and difficulty. With regard to difficulty there are four kinds of problems:

- Basic Exercises
- True/False
- More Difficult Problems
- Challenge Problems

You should think of basic exercises as problems that are most like problems that appear on quizzes, and the true/false and difficult problems as those that are most like problems that appear on exams (several of these problems and exercises have appeared on quizzes and exams in the past). The challenge problems are for those of who want to think a bit harder, and are most like problems that may appear as extra credit on exams.

Many of the problems and exercises here are based on those that appear in *Linear Algebra and Its Applications* by David Lay, Steven Lay and Judi McDonald. If a problem in this workbook is identical to one that appears there, it is by accident or by coincidence. Regardless, we are in debt to this seminal text on undergraduate linear algebra.

Problems that require a computer or a calculator are labelled with the symbol 🖩. Otherwise, the expectation is that you should be able to solve the problem by hand.

Solutions to these problems will never be made public, except during a given semester when a problem is given in an assignment. Also, during the semester a student of the course is welcome to request a partial solution any problem via our course public forum (usually Piazza).

I'd like to acknowledge folks who have contributed problems that appear in this workbook, including Ed Chien, Mark Crovella, Vishesh Jain, Ieva Sagaitis, and Abhinit Sati. I also include David C. Lay, Steven R. Lay, and Judi J. McDonald by proxy.

Lastly, even if it's not explicitly stated, **you should always show your work and justify your answer for every problem.** Happy exercising/problem solving!

Chapter 1

Linear Systems

Determine the coefficient matrix and augmented matrix for each of the following linear systems.

1.

$$\begin{aligned} 2x_1 - 6x_2 &= -4 \\ 6x_1 + 8x_2 &= -7 \end{aligned}$$

2.

$$\begin{aligned} x_1 + 8x_2 + 7x_3 &= -1 \\ 4x_1 - 9x_2 &= 8 \\ 7x_1 - 7x_2 - 3x_3 &= 10 \end{aligned}$$

3.

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= 2 \\ 2x_1 - 3x_2 - 5x_3 &= 2 \\ -2x_1 + 2x_2 + 7x_3 &= -1 \end{aligned}$$

4.

$$\begin{aligned} -8x_1 + 6x_2 + x_3 + 7x_4 + 10x_5 &= 8 \\ -4x_2 - 4x_3 - x_4 + x_5 &= -8 \\ 7x_1 + x_2 - 2x_3 - 8x_4 + 7x_5 &= 9 \\ -4x_1 - 7x_2 + 5x_3 + 2x_4 - 9x_5 &= -4 \end{aligned}$$

For each of the following linear systems, verify that the given point s is a solution.

5.

$$\begin{aligned} x_1 - 2x_2 + x_3 - 2x_4 &= -9 \\ x_1 - x_2 - x_3 - 2x_4 &= -10 \\ -3x_1 + 8x_2 - 6x_3 + 4x_4 &= 21 \\ 2x_2 - 7x_3 + 7x_4 &= 13 \end{aligned}$$

$$s = (1, 3, 2, 3)$$

6.

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + 4x_4 &= -20 \\ x_1 + 2x_2 - 6x_3 - 4x_4 &= 26 \\ x_1 - 4x_3 - x_4 &= 5 \end{aligned}$$

$$s = (10, 8, 2, -3)$$

Demonstrate that each of the following linear systems has a unique solution. Additionally, determine the solution.

7.

$$\begin{aligned} 3x_1 + 6x_2 &= -15 \\ -x_1 + x_2 &= 5 \end{aligned}$$

8.

$$\begin{aligned} 2x + 4y &= 29 \\ -6x - 10y &= -75 \end{aligned}$$

9.

$$\begin{aligned} x + 3y - 3z &= -2 \\ 2x + 7y + 7z &= -5 \\ x - 2y - z &= 3 \end{aligned}$$

10.

$$\begin{aligned} 3x - 4y + 3z &= -9 \\ 6x + 7y - 3z &= 0 \\ x + 10z &= -21 \end{aligned}$$

11.

$$\begin{aligned} -3x_1 + 6x_3 &= 12 \\ 3x_1 + 3x_2 - 12x_3 &= -21 \\ 2x_1 - 6x_2 + 10x_3 &= 16 \end{aligned}$$

12.

$$\begin{aligned} -3x_1 + 6x_2 &= 9 \\ 3x_1 - 8x_2 &= -9 \\ 2x_2 + 6x_3 &= 18 \end{aligned}$$

13.

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= -7 \\ -x_1 + 3x_2 + 2x_3 &= 10 \\ 2x_1 - 6x_2 - 3x_3 &= -18 \end{aligned}$$

14.

$$\begin{aligned} -3x_1 - 6x_3 + 9x_4 &= -6 \\ -x_1 + 2x_2 - 3x_4 &= 8 \\ -2x_1 - 9x_2 - 19x_3 + 45x_4 &= -61 \\ x_1 - 3x_2 + 5x_3 - 9x_4 &= 5 \end{aligned}$$

For each of the following matrices, apply the given row operations from top to bottom.

15.

$$\begin{bmatrix} -2 & 7 & 9 \\ 7 & -9 & -9 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\begin{aligned} &R_1 \leftarrow -5R_1 \\ &R_1 \leftarrow R_1 - 4R_2 \\ &R_2 \leftarrow R_2 + 3R_1 \end{aligned}$$

16.

$$\begin{bmatrix} -4 & 1 & 3 \\ -7 & 0 & 5 \\ -4 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} &R_3 \leftarrow R_3 - 3R_2 \\ &R_3 \leftarrow R_3 + 3R_1 \\ &R_1 \leftrightarrow R_3 \end{aligned}$$

17.

$$\begin{bmatrix} 9 & 5 & -7 & -5 & -9 \\ 5 & -7 & 1 & -2 & -9 \\ 5 & 1 & -10 & 6 & -5 \\ 5 & 7 & -5 & 2 & 1 \end{bmatrix}$$

$$R_4 \leftarrow -R_4$$
$$R_2 \leftarrow R_2 - 2R_3$$
$$R_2 \leftarrow R_2 - 5R_4$$
$$R_3 \leftarrow R_3 + 3R_4$$
$$R_3 \leftrightarrow R_2$$

Determine a general form solution from each of the following matrices. That is, determine a general form solution for a linear system whose augmented matrix is row equivalent to the given matrix.

18.

$$\begin{bmatrix} 1 & -3 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

19.

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

20.

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 3 \\ 0 & 1 & -3 & 2 & -9 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

21.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 1 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine three particular solutions of the linear system underlying each of the following matrices. That is, determine three particular solutions of a linear system whose augmented matrix is row equivalent to the given matrix.

22.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

23.

$$\begin{bmatrix} 1 & 0 & 0 & -3 & -6 \\ 0 & 1 & 0 & 5 & 5 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix}$$

24.

$$\begin{bmatrix} 1 & 1 & 0 & -3 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine the row-reduced echelon form of each of the following matrices. You must include the intermediate matrices and row operations used.

25.

$$\begin{bmatrix} 5 & 2 & 9 \\ 7 & 3 & 12 \\ 2 & 1 & 3 \end{bmatrix}$$

26.

$$\begin{bmatrix} 0 & 1 & -6 & -2 \\ 1 & 1 & -3 & -6 \\ -2 & -4 & 18 & 16 \end{bmatrix}$$

27.

$$\begin{bmatrix} 2 & 3 & -13 & -8 \\ 1 & 1 & -4 & -1 \\ 1 & 0 & 1 & 5 \end{bmatrix}$$

28.

$$\begin{bmatrix} 1 & -1 & -2 & 1 \\ -1 & 2 & 4 & 0 \\ 2 & -3 & -6 & 2 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

Determine a general form solution for each of the following linear systems by first determining the row-reduced echelon form of its augmented matrix. If this system has no solutions then write "no solution".

29.

$$\begin{aligned} x - 2y - 7z &= -3 \\ 2x + y + 6z &= 14 \\ x + y + 5z &= 9 \end{aligned}$$

30.

$$\begin{aligned} x_1 + x_2 - x_3 &= -9 \\ x_2 - 2x_3 &= -1 \\ x_1 + x_2 &= -10 \end{aligned}$$

31.

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 6 \\ -2x_1 + 6x_2 - 11x_3 + 7x_4 &= -8 \\ 5x_1 - 10x_2 + 25x_3 + 3x_4 &= 30 \end{aligned}$$

32.

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 0 \\ -x_1 + 6x_2 + x_3 &= 4 \\ 23x_2 + 5x_3 &= 9 \end{aligned}$$

33.

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= -4 \\ x_1 + 6x_2 + 7x_3 &= -13 \\ -2x_1 - 12x_2 - 14x_3 &= 25 \end{aligned}$$

34.

$$\begin{aligned} -x - 2y - z &= -4 \\ x + 2y &= 3 \\ -2x - 4y + z &= -7 \end{aligned}$$

35.

$$\begin{aligned} x_1 - 6x_2 + x_4 &= 2 \\ 2x_1 - 12x_2 + x_3 - 4x_4 &= 7 \\ x_1 - 6x_2 - 2x_3 + 13x_4 &= -4 \end{aligned}$$

36.

$$\begin{aligned} 88x_1 + 61x_2 + 8x_3 + 61x_4 &= -68 \\ 37x_1 + 14x_2 + 72x_3 + 41x_4 &= -460 \\ 57x_1 + 59x_2 + 69x_3 + 75x_4 &= -333 \\ 92x_1 + 60x_2 + 28x_3 + 72x_4 &= -192 \end{aligned}$$

37.

$$\begin{aligned} x_2 + x_3 + x_4 + 3x_5 - x_6 + 2x_7 &= 34 \\ x_3 - x_4 + 4x_5 &= -14 \\ 2x_2 - 4x_3 + 8x_4 - 18x_5 - x_6 &= 110 \\ 3x_3 - 3x_4 + 12x_5 + 2x_7 &= -18 \\ x_2 + 4x_3 - 2x_4 + 15x_5 - x_6 + 7x_7 &= 52 \end{aligned}$$

38.

$$\begin{aligned} 8x_1 + 3x_2 - 3x_3 &= -4 \\ -3x_1 + 8x_2 + 5x_3 &= -4 \\ 6x_1 + 7x_2 - 3x_3 &= -3 \\ 8x_1 + x_2 - x_3 &= -4 \end{aligned}$$

Determine a sequence of elementary row operations from the left matrix to the right matrix. You must including the intermediate matrices and row operations used.

39.

$$\begin{bmatrix} -4 & -7 & 4 \\ 3 & 8 & 2 \\ -10 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 3 & 8 & 2 \\ -14 & -6 & -5 \\ -10 & 1 & -9 \end{bmatrix}$$

40.

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 2 & 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 0 & 6 \\ 4 & 4 & 2 & 2 \\ 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

41. Elementary row operations cannot change the solution set of a linear system.

42. There is a linear system with exactly three solutions.

43. If A is the augmented matrix of an inconsistent linear system, and B is a matrix such that $A \sim B$ (that is, A and B are row equivalent), then B is the augmented matrix of an inconsistent linear system.

44. If A is the augmented matrix of an inconsistent system, then A has a pivot in its rightmost column.

45. If $A \sim B$ and $A \sim C$ and B and C are in reduced echelon form, then $B = C$.

46. There is a unique sequence of row operations that reduces a matrix to reduced echelon form.

47. If a general form solution of a linear system has a free variable, then the system has infinitely many solutions.
48. A matrix may have different pivot positions depending on the sequence of row operations used to attain a matrix in echelon form.
49. Every matrix has a unique echelon form.
50. A linear system over 3 variables and 2 equations must be consistent.
51. If the coefficient matrix of a linear system as more rows than columns, then the system has infinitely many solutions.
52. If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.
53. A *consistent* linear system whose augmented matrix is a 207×209 matrix has infinitely many solutions.
54. A *consistent* linear system whose coefficient matrix is square has infinitely many solutions.
55. If the rightmost column of the augmented matrix of a linear system is a pivot column, then the system is inconsistent.
56. If a system of linear equations of linear equations has infinitely many solutions, then it must have more unknowns than equations.
57. For any nonzero real values a and b, the matrix $\begin{bmatrix} a & 2a \\ b & 3b \end{bmatrix}$ has a pivot in every column and every row.
58. A system of linear equations with 3 variables and 4 equations cannot be consistent.

More Difficult Problems

59. Consider the following linear system.

$$\begin{aligned} 4x_2 + x_3 &= 16 \\ 9x_1 - 20x_2 - 8x_3 &= -71 \\ 3x_1 - 8x_2 - 3x_3 &= -29 \end{aligned}$$

   (a) Determine the RREF of the augmented matrix of the above linear system.
   (b) Determine a general form solution for the above linear system.
   (c) Find a solution to the linear system below with the following properties:
      i. the solution consists entirely of integer values;
      ii. the values in the solution are relatively prime, i.e., it's not possible to divide the solution by a number to get another integer solution. So $(2, 4, 6)$ does not satisfy this property but $(1, 2, 3)$ does.[1]

[1] This is the process we would use to find a valid solution to the problem of balancing chemical equations.

60. Determine all the 2×2 matrices in echelon form whose entries are either 0 or 1. Mark which ones are in reduced echelon form.

61. Determine every 3×3 matrix in reduced echelon form with at least two pivot positions whose entries are either 0 or 1.

62. Determine a linear system over three variables such that (a, b, c) is a solution exactly when the cubic function $f(x) = ax^3 + bx^2 + c$ intersects the points $(-1, 4)$, $(1, 5)$ and $(2, 10)$. You do not need to solve the linear system.

63. Determine the slope-intercept form of the line equation which defines the intersection of the plane

$$2x + 3y + 3z = 6$$

with the xy-plane.

64. For what values of the coefficient h is the following system inconsistent?

$$\begin{aligned} x + 4y &= -1 \\ 3x - hy &= 7 \end{aligned}$$

Is there a value of h for which the above system has infinitely many solutions? Justify your answer.

65. Consider the following linear system with two unknown coefficients h and k.

$$\begin{aligned} hx + 2y &= 1 \\ 3x + 9y &= k \end{aligned}$$

(a) Determine values of h and k so that the above linear system has no solutions.

(b) Determine values of h and k so that the above linear system has exactly one solution.

(c) Determine values of h and k so that the above linear system has infinitely many solutions.

66. Consider the following general form solution

$$\begin{aligned} x_1 &= 3x_2 - 2x_3 \\ x_2 &\quad \text{is free} \\ x_3 &\quad \text{is free} \\ x_4 &= 2 + x_3 \end{aligned}$$

(a) Determine another general form solution which describes the same solution set, but for which x_3 is basic and x_4 is free.

(b) Determine an RREF which yields the general form solution from the previous part.

67. Consider the following general form solution.

$$\begin{aligned} x_1 &= -6 + 6x_3 + 2x_5 \\ x_2 &= 4 + 4x_3 + 6x_5 \\ x_3 &\text{ is free} \\ x_4 &= -4 + 5x_5 \\ x_5 &\text{ is free} \end{aligned}$$

Determine a general form solution that describes the same solution set but in which x_1 is free.

68. Suppose you're investigating a claim that, out of five competing car companies who purchase engine parts from the same suppliers, one purportedly *overestimated* their total spending by \$1,300,000. Companies are required to report their total spending, and you've been able to determine how many units ($\times$ 10,000) of each part that each company has purchased. You haven't been unable to determine how much each item costs per unit (it's an industry secret), but you can assume that each company pays the same amount per unit. Given the following data, which company is falsifying their records? Justify our answer. The total amount spent by each company in the table below is multiplied by \$100,000.

Co.	Part 1	Part 2	Part 3	Part 4	Part 5	Total Spent
A	15	3	24	46	182	1013
B	5	3	14	25	100	552
C	15	3	24	46	188	1038
D	5	0	5	10	40	225
E	15	3	26	47	190	1056

69. Suppose you're given a system of linear equations with three variables and two equations, and that $(4, 1, 0)$ and $(2, 4, 1)$ are solutions to this system. You are also given that the two equations define distinct planes in $\mathbb{R}^3$. Determine the RREF of the augmented matrix for this system.

70. Repeat the previous problem with the points $(-3, 0, 1)$ are $(-3, 1, 1)$.

71. Consider an arbitrary system of linear equations with n unknowns and m equations. Further suppose that

- it has a unique solution;
- it has at least as many equations as unknowns ($m \geq n$).

Write down an expression in terms of m and n for the number of all-zero rows which appear in the rwo-reduced echelon form of its augmented matrix. Justify your answer.

Challenge Problems

72. Consider the following pair of linear systems.

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

$$\begin{aligned} ax + by - cz &= 0 \\ dx + ey - fz &= 0 \end{aligned}$$

(a) Demonstrate that if the first system has a solution, then so does the second one.

(b) Give **nonzero** values to a through f such that the second system has a solution, but the first does not. Present your solution as an augmented matrix for the first system, i.e., of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

73. Consider the following pair of linear systems.

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 4x_1 + x_2 &= 21 \end{aligned}$$

$$\begin{aligned} 10x_3 + 2x_4 &= x_1 \\ -8x_3 + 9x_4 &= x_2 \end{aligned}$$

(a) Solve the first system of linear equations (in x_1 and x_2) and write down the augmented matrix of the second system with the solutions of x_1 and x_2 substituted in.

(b) Determine the augmented matrix of a *single* system of linear equations *with all four equations* in the variables x_1, x_2, x_3, x_4. Describe the relationship between this matrix and the one in the previous part.

74. Determine what must hold of a, b, c, d so that the following linear system has exactly one solution.

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

75. Determine an inconsistent linear system in 3 variables such that every pair of equations is consistent. That is, give values for a through l in the system

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

such that the system is inconsistent, but each pair of equations forms a consistent system. Present your solution to each part as the an augmented matrix, i.e., of the form

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

(a) Achieve this with no more than 5 nonzero values for a through l.

(b) Achieve this with all nonzero values for a through l.

76. Consider an arbitrary linear system in three variables:

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

Show that if (s_x, s_y, s_z) and (t_x, t_y, t_z) are solutions to the above system, then

$$\left(\frac{s_x + t_x}{2}, \frac{s_y + t_y}{2}, \frac{s_z + t_z}{2}\right)$$

is also a solution. Describe why $(s_x, s_y, s_z) \neq (t_x, t_y, t_z)$ implies the system has infinitely many solutions.

77. One way to describe a line in 3 dimensions is to use a parameter t. That is, after fixing values a_x, b_x, a_y, b_y, a_z, and b_z, a line can be described as all points of the form

$$(a_x t + b_x, a_y t + b_y, a_z t + b_z)$$

for any real value of t. Give a pair of 3 dimensional linear equations (each of which represents a plane in $\mathbb{R}^3$) whose intersection is exactly the line whose points are defined by the above parametric form. *Hint:* The line above can be thought of as a system in the variables x, y, z, and t, e.g., one of its equations is $x - a_x t = b_x$. Write t in terms of x and substitute this value for t into the other equations.

78. There are several ways to define row equivalence. For example, suppose we restrict the replacement rule to only allow operations of the form

$$R_i \leftarrow R_i + R_j$$

That is, we can only add one row to another, without doing any scaling of that row. We call this an *addition operation*.

Demonstrate that two matrices A and B are row equivalent if there is a sequence of addition and scaling operations which transform A to B. In particular, addition and scaling operations can simulate replacement and exchange operations.

Chapter 2

Vector Equations and Spans

Basic Exercises

Compute the following linear combinations of vectors.

1.

$$7\begin{bmatrix}-4\\-6\end{bmatrix}+8\begin{bmatrix}3\\-8\end{bmatrix}$$

2.

$$4\begin{bmatrix}-1\\1\\10\end{bmatrix}-5\begin{bmatrix}-6\\3\\7\end{bmatrix}+9\begin{bmatrix}8\\9\\1\end{bmatrix}$$

3.

$$7\begin{bmatrix}2\\-3\\-8\\9\end{bmatrix}+3\begin{bmatrix}2\\-3\\-4\\-3\end{bmatrix}-4\begin{bmatrix}5\\-1\\5\\-7\end{bmatrix}-2\begin{bmatrix}-2\\-9\\-2\\-10\end{bmatrix}$$

For each of the following linear systems, determine an equivalent vector equation, i.e., one that has the same solution set as the given linear system.

4.

$$\begin{aligned}4x_1+9x_3-10x_4&=4\\10x_1+4x_2-6x_3+x_4&=-8\end{aligned}$$

5.

$$\begin{aligned}-8x_1+8x_2-5x_3&=-4\\-2x_2-8x_3&=10\\-10x_1+6x_2-2x_3&=0\end{aligned}$$

6.

$$
\begin{aligned}
8x_1 + 6x_2 - 9x_3 &= -5 \\
4x_1 + 3x_2 + 9x_3 + 2x_4 &= -1 \\
4x_1 + 5x_2 &= 9 \\
3x_2 + 8x_3 - 3x_4 &= 2
\end{aligned}
$$

Determine a vector that is in span$\{\mathbf{v}_1, \mathbf{v}_2\}$ but not in span$\{\mathbf{v}_1\}$ or span$\{\mathbf{v}_2\}$, if possible. Justify your answer. In particular, if it is not possible, then explain why.

7.

$$\mathbf{v}_1 = \begin{bmatrix} -12 \\ -8 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

8.

$$\mathbf{v}_1 = \begin{bmatrix} -7 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 9 \\ -7 \end{bmatrix}$$

Determine a general form solution for each vector equation. If the given equation has no solutions then write "no solution".

9.

$$x_1 \begin{bmatrix} -2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ -9 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

10.

$$x_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -9 \\ -13 \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \\ -16 \end{bmatrix}$$

11.

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ 21 \\ 10 \end{bmatrix} = \begin{bmatrix} 16 \\ -38 \\ -12 \end{bmatrix}$$

For each of the following collections of vectors, determine if the vector $\mathbf{v}_1$ is in the span of the remaining vectors. If it is, determine the corresponding dependence relation, i.e., write $\mathbf{v}_1$ as a linear combination of the remaining vectors.

12.

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -3 \\ -20 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$$

13.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

14.

$$\mathbf{v}_1 = \begin{bmatrix} 42 \\ -23 \\ 98 \\ 11 \\ -87 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 10 \\ -33 \\ -5 \\ -30 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -47 \\ -1 \\ -2 \\ -25 \\ 24 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 31 \\ -34 \\ 11 \\ 39 \\ 25 \end{bmatrix} \qquad \mathbf{v}_5 = \begin{bmatrix} -14 \\ 22 \\ 12 \\ 42 \\ 3 \end{bmatrix}$$

15.

$$\mathbf{v}_1 = \begin{bmatrix} 87 \\ -19 \\ -24 \\ -61 \\ -79 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 10 \\ -33 \\ -5 \\ -30 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -47 \\ -1 \\ -2 \\ -25 \\ 24 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 31 \\ -34 \\ 11 \\ 39 \\ 25 \end{bmatrix} \qquad \mathbf{v}_5 = \begin{bmatrix} -14 \\ 22 \\ 12 \\ 42 \\ 3 \end{bmatrix}$$

For each of the following pairs of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, determine a linear equation whose point set is the span$\{\mathbf{v}_1, \mathbf{v}_2\}$. The linear equation you determine should have relatively prime integer coefficients (i.e., it should not be possible to divide the equation by an integer value and get a new equation with integer coefficients).

16.

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

17.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

18.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

19.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

20. For any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, there is a vector $\mathbf{u}$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{u} = \mathbf{0}$.

21. For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\mathbf{v}_1 \in \text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$, then $\mathbf{v}_2 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

22. The span of any two distinct nonzero vectors in $\mathbb{R}^3$ is a plane.

23. For any vector $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$.

More Difficult Problems

24. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Write vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ such that $\mathbf{v}_3$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{v}_4$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

25. Determine a vector with integer entries that appears in both of the following spans.

$$\text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} \qquad \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}$$

26. Repeat the previous problem with the following spans.

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \qquad \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}\right\}$$

Challenge Problems

27. Determine a linear equation in three variables whose point set is exactly

$$\left\{\mathbf{v} + \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \text{ where } \mathbf{v} \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}\right\}$$

In other words, every point in the point set of the equation can be expressed as the sum of $[2 \;\; (-3) \;\; 2]^T$ and a vector in the span of $[1 \;\; 1 \;\; (-5)]^T$ and $[0 \;\; 1 \;\; (-1)]^T$.

28. Consider the following linear equation.

$$x + y + z = 5$$

The plane represented by this equation does not include the origin; such a plane is called *affine*. Determine vectors $\mathbf{b}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ such that every point in the plane represented by the above equation (that is, every point (a, b, c) such that $a + b + c = 5$) can be written as $\mathbf{b} + \mathbf{v}$ where $\mathbf{v}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, i.e.,

$$\{(x, y, z) : x + y + z = 5\} = \{\mathbf{b} + \mathbf{v} : \mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}\}.$$

Chapter 3

Matrix Equations

Compute the matrix-vector multiplication $A\mathbf{v}$ where A and $\mathbf{v}$ are given below. If it is not possible to multiply A with $\mathbf{v}$, then explain why.

1.

$$A = \begin{bmatrix} -10 & 6 & 2 & 8 \\ 1 & 3 & 4 & 5 \\ 0 & -2 & 0 & -9 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 4 \\ -5 \\ 3 \\ 1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 6 & 1 & -8 & -3 \\ 5 & 0 & -9 & -4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

Determine a general form solution for the matrix equation $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given below. If this equation has no solutions then write *NO SOLUTION*.

3.

$$A = \begin{bmatrix} 1 & -3 & 1 & -9 \\ 1 & -2 & 0 & -5 \\ -3 & 8 & -1 & 19 \\ -2 & 4 & 0 & 10 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ -18 \\ -14 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & -7 & 1 \\ -5 & -4 & -14 & -3 & 24 & -8 \\ 0 & -2 & -2 & -3 & 21 & 3 \\ -6 & -3 & -15 & -2 & 11 & -8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 7 \\ -16 \\ -20 \\ -27 \end{bmatrix}$$

For each of the following matrices, Determine if its columns have full span, i.e., given a matrix in $\mathbb{R}^{m \times n}$, determine if the columns span $\mathbb{R}^m$.

5.

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

6.

$$\begin{bmatrix} 1 & -5 & 4 \\ -1 & 6 & -3 \\ -2 & 13 & -7 \end{bmatrix}$$

7.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ -1 & -1 & 2 & 9 \\ -1 & -2 & 1 & 1 \\ 0 & 2 & 6 & 36 \end{bmatrix}$$

8.

$$\begin{bmatrix} 8 & 3 & 8 \\ 8 & -4 & 4 \\ -2 & 1 & -5 \\ -5 & -7 & -10 \\ -8 & 9 & 1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

9. For $A \in \mathbb{R}^{m\times n}$ where $m > n$, it is not possible for the columns of A to span $\mathbb{R}^m$.

10. For $A \in \mathbb{R}^{m\times n}$, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then it is not possible for the columns of A to span $\mathbb{R}^m$.

11. For $A \in \mathbb{R}^{m\times n}$, if $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the columns of A span $\mathbb{R}^m$.

12. For $A \in \mathbb{R}^{m\times n}$, if $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $\{\mathbf{v} : A\mathbf{v} = \mathbf{0}\} = \mathbb{R}^n$.

13. For matrices A and B in $\mathbb{R}^{m\times n}$, let $[A\ B] \in \mathbb{R}^{m\times 2n}$ be the matrix obtained by horizontally stacking A and B. If $A\mathbf{x} = \mathbf{b}$ is consistent and $B\mathbf{x} = \mathbf{b}$ is consistent, then so is $[A\ B]\mathbf{x} = \mathbf{b}$.

14. For matrices A and B in $\mathbb{R}^{m\times n}$, if $[A\ B]\mathbf{x} = \mathbf{b}$ is consistent then $A\mathbf{x} = \mathbf{b}$ is consistent or $B\mathbf{x} = \mathbf{b}$ is consistent.

15. For $A \in \mathbb{R}^{m\times n}$, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then A does not have a pivot position in every column.

16. If the columns of $A \in \mathbb{R}^{m\times n}$ span all of $\mathbb{R}^m$, then the reduced echelon form of A has only 0s and 1s.

17. For any matrix A in $\mathbb{R}^{10\times 15}$ and any $\mathbf{b}$ in $\mathbb{R}^{10}$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

18. For any 20×24 matrix A, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

19. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then it has infinitely many solutions.

More Difficult Problems

20.

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

For each of the following shapes, determine a concrete matrix A so that the above equation holds, where $\blacksquare$ represents a nonzero entry.

(a)

$$\begin{bmatrix} \blacksquare & 0 & 0 & 0 \\ 0 & \blacksquare & 0 & 0 \\ 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

(b)

$$\begin{bmatrix} \blacksquare & 0 & 0 & 0 \\ \blacksquare & \blacksquare & 0 & 0 \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

(c)

$$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & 0 & 0 \\ \blacksquare & 0 & 0 & 0 \end{bmatrix}$$

21. Any three distinct points in the plane define a *unique* quadratic equation. Given three points (x_1, y_1), (x_2, y_2) and (x_3, y_3), the equation $y = ax^2 + bx + c$ that passes through these three points is given by the solution to the following matrix equation.

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Use it to determine the unique quadratic equation which passes through the points $(2, 13)$, $(3, 25)$ and $(-2, 5)$.

22. Determine the RREF of the following matrix in terms of x, y, assuming $x \neq 1$.

$$\begin{bmatrix} x^2 & x & 1 & y \\ 1 & 1 & 1 & 1 \\ \left(\frac{x+1}{2}\right)^2 & \frac{x+1}{2} & 1 & \frac{y+1}{2} \end{bmatrix}$$

Hint: Don't try to row reduce it. Think in terms of polynomial interpolation.

23. Let A be a matrix with n rows and 6 columns. Each row of A contains the **unweighted** percentage scores (out of 100) of one student on 4 homework assignments (columns 1 through 4) a midterm exam (column 5) and a final exam (column 6).

$$\begin{matrix} H_1 & H_2 & H_3 & H_4 & M & F \end{matrix}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & p_{n4} & p_{n5} & p_{n6} \end{bmatrix}$$

All homework assignments are worth the same amount. Let T denote the linear transformation implemented by this matrix.

(a) Suppose that homework assignments account for **50** percent of the final grade, the midterm exam accounts for **20** percent and the final exam accounts for **30** percent. Find a vector $\mathbf{v}$ such that $T(\mathbf{v})$ is the vector whose i^{th} entry is the final percentage grade of the i^{th} student. For example, if the i^{th} student recieved 90 percent on every homework assignment, 85 percent on the midterm, and 92 percent on the final, then the i^{th} entry of the output vector should be $90 * 0.5 + 85 * 0.2 + 92 * 0.3$.

(b) Find a vector $\mathbf{v}$ such that $T(\mathbf{v})$ is the vector whose i^{th} entry is the unweighted homework grade for student i. For the same example as above, the i^{th} entry would be 90.

24. Consider the following matrix and vector.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{e}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) Explain why the columns of B span $\mathbb{R}^6$.

(b) Determine a solution for the equation $B\mathbf{x} = \mathbf{e}_6$.

Challenge Problems

25. Consider the following matrix and vector.

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{e}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Determine a solution for the equation $C\mathbf{x} = \mathbf{e}_6$.

Chapter 4

Linear Independence

Basic Exercises

Determine if the following collections of vectors are linearly dependent. If they are write a dependence relation, i.e., determine linear combination of the given vectors which sums to $\mathbf{0}$.

1.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ -8 \end{bmatrix}$$

3.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 7 \\ -2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -6 \\ 15 \\ -3 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

4.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors such that the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ is row equivalent to matrix given below. Determine a dependence relation for the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

5.

$$\begin{bmatrix} 1 & 1 & -3 & 5 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6.

$$\begin{bmatrix} 2 & -3 & 3 & 8 \\ 0 & -2 & -2 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

7. For $A \in \mathbb{R}^{m\times n}$, if the columns of A are linearly dependent, then they do not span $\mathbb{R}^m$.

8. If the columns of A are linearly independent, then the reduced echelon form of A has only 0s and 1s.

9. For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, if $\mathbf{v}_1 \in \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_3\}$ then $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3\}$ is a linearly dependent set.

10. For any $m \times n$ matrix A, if $m > n$ then the columns of A must be linearly independent.

11. For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

12. For any matrix A in $\mathbb{R}^{m\times n}$ and distinct vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, if $A\mathbf{v}_1 = \mathbf{0}$ and $A\mathbf{v}_2 = \mathbf{0}$, then the columns of A are linearly dependent.

13. For any matrix A in $\mathbb{R}^{m\times n}$, if the columns of A^T are linearly dependent, then the columns of A do not span $\mathbb{R}^m$.

14. If the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent, then $\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

15. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is the set $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_2, \mathbf{v}_3 + \mathbf{v}_2\}$.

More Difficult Problems

16. Consider four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ in $\mathbb{R}^4$ such that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & -2 \\ 0 & 4 & -4 & 12 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Determine if $\mathbf{v}_4 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If so, express $\mathbf{v}_4$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

   (b) Determine if vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ linearly independent. If so, justify your answer. If not, determine a dependence relation for these vectors.

17. Determine all values of h for which the following set of vectors is linearly dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 9 \\ -3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -2 \\ h \\ -6 \end{bmatrix}$$

18. Determine three *nonzero* vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that

- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and
- $\mathbf{v}_1$ cannot be written as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

19. Consider the following matrix A and vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ -1 & 0 & -3 & 2 \\ 2 & 3 & 12 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 11 \end{bmatrix}$$

(a) Determine if $\mathbf{b} \in \operatorname{Col} A$, i.e., if the vector $\mathbf{b}$ is in the span of the columns of A. If it is, then express $\mathbf{b}$ as a nontrivial linear combination of the columns of A (you should use $\mathbf{a}_i$ to denotes the i^{th} column of A). If it is not, then justify your answer.

(b) Determine if the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent. If it is, write a dependence relation for this set of vectors. If it is not, then justify your answer.

20. Consider the following matrix, presented as a SymPy array.

```
import sympy
a = sympy.Matrix(
    [[ 13,  -19,  19,  16,   5,   1,  10,   5,  15],
     [-11,   -7, -10,   7,   2,  -8, -10, -19,   6],
     [ -5,   10,  -7,   2,  -8,   2, -15, -16, -11],
     [ 17,  -13,   9,  13,  19,   8,  -3,  -9,   0],
     [  9,  -18,   5,   1,   4,  14,   9,   8,  -4],
     [  8,   14,  17,   5,  -6,   7, -13,   2,  12],
     [ 18,   12,  -7,   2, -10,  15, -12,   1, -12],
     [ 19,  -12,   1, -16,   2,  -6,  -4,  17,  15],
     [-19,   -6, -16, -20, -20,  -3,   7,   3,  14],
     [  6,    8, -15,   5,  -5,   8, -14,   5, -19]])
```

(a) Determine if the columns of this matrix span all of $\mathbb{R}^{10}$. Justify your answer.

(b) Determine if the columns of this matrix linearly independent. Justify your answer.

21. Consider the following collection of vectors.

$$\mathbf{v}_1 = \begin{bmatrix} -10 \\ -6 \\ 2 \\ 0 \\ 8 \\ -3 \\ -4 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -8 \\ 5 \\ -2 \\ 7 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -20 \\ -12 \\ 4 \\ 0 \\ 16 \\ -6 \\ -8 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 1 \\ 3 \\ -1 \\ 10 \\ 7 \\ 4 \end{bmatrix}$$

$$\mathbf{v}_5 = \begin{bmatrix} -2 \\ -10 \\ 10 \\ -5 \\ -3 \\ -10 \\ -5 \end{bmatrix} \qquad \mathbf{v}_6 = \begin{bmatrix} -2 \\ 3 \\ -9 \\ -2 \\ -9 \\ -2 \\ 7 \end{bmatrix} \qquad \mathbf{v}_7 = \begin{bmatrix} -13 \\ 25 \\ -22 \\ 13 \\ 24 \\ 41 \\ 15 \end{bmatrix}$$

(a) Determine the first vector (i.e., the vector with the smallest index) which can be written as a linear combination as of the vectors which precede it. Express this vectors as a linear combination of the vectors which precede it. You should write your final solution using the vector names of the form $\mathbf{v}_i$.

(b) Determine a dependence relation for the entire set of vectors with the following properties:

- the coefficients of the dependence relation are relatively prime integers;
- the number of nonzero coefficients is maximum, i.e., there is no other dependence relation with a greater number of nonzero coefficients.

22. Consider three arbitrary vectors in $\mathbb{R}^3$.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and suppose that they are linearly independent.

(a) Determine the maximum number of entries of these vectors which can be 0.

(b) Determine the minimum number of entries which can be zero 0.

Provide an example in each case.

23. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors and that

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 + \mathbf{v}_3 \\ \mathbf{u}_2 &= -2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{u}_3 &= -3\mathbf{v}_1 - \mathbf{v}_2 - 6\mathbf{v}_3 \end{aligned}$$

Determine a dependence relation with integer weights for the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

24. Consider the following vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -6 \\ 0 \\ 7 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \\ -1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

(a) Determine if $\mathbf{v}_1$ is in span$\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Justify your answer. In particular, if $\mathbf{v}_1$ is in span$\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, then write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

(b) Determine if the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them (that is, write the zero vector $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$).

(c) Determine if the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them.

25. Repeat the previous problem with the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 4 \\ 3 \\ 7 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Challenge Problems

26. Determine all values of h for which the following set of vectors is linearly dependent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ h \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 13 \\ h \\ -1 \end{bmatrix}$$

Chapter 5

Linear Transformations

Basic Exercises

For each of the following linear transformations, its domain, codomain, and the matrix that implements it (i.e., determine the matrix A such that the given transformation is $\mathbf{x} \mapsto A\mathbf{x}$).

1.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_2 \\ -2x_1 - x_3 \\ x_1 + x_3 \end{bmatrix}$$

2.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \\ x_3 + x_4 \end{bmatrix}$$

3.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 5x_1 + 7x_2 + 10x_3 \\ 2x_1 - x_2 + 4x_3 \\ -3x_2 \end{bmatrix}$$

4.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} -9x_1 + 4x_2 + 5x_3 - 3x_4 + 2x_5 \\ 3x_1 + 8x_2 + 5x_3 - 5x_4 + 7x_5 \\ -x_1 + 5x_2 - x_3 + 4x_4 + 8x_5 \end{bmatrix}$$

5.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} -2x_2 + x_3 + x_4 \\ x_1 + x_3 \\ -3x_3 - 3x_4 \\ 9x_4 \end{bmatrix}$$

Let T be a linear transformation with the following input-output behavior.

$$T(\mathbf{v}_1) = \begin{bmatrix} -6 \\ 3 \\ -2 \\ -10 \end{bmatrix} \qquad T(\mathbf{v}_2) = \begin{bmatrix} -5 \\ 1 \\ -2 \\ 9 \end{bmatrix} \qquad T(\mathbf{v}_3) = \begin{bmatrix} 8 \\ -7 \\ 6 \\ 6 \end{bmatrix}$$

Compute the following.

6. $T(\mathbf{v}_1 + \mathbf{v}_2)$

7. $T(-3\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3)$.

Let T be a linear transformation with the following input-output behavior.

$$T\left(\begin{bmatrix} -3 \\ 9 \\ 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} -9 \\ -5 \\ 0 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 9 \\ -1 \\ -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}$$

Compute the following.

8.

$$T\left(\begin{bmatrix} 30 \\ 0 \\ -7 \\ 22 \end{bmatrix}\right)$$

9.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

Determine the matrix that implements the linear transformation T, given its input-output behavior.

10.

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

11.

$$T\left(\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

12.

$$T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right)=\begin{bmatrix}3\\-3\\4\end{bmatrix}\qquad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}8\\0\\2\end{bmatrix}\qquad T\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

13.

$$T\left(\begin{bmatrix}-7\\-3\end{bmatrix}\right)=\begin{bmatrix}-4\\3\\4\\0\\1\end{bmatrix}\qquad T\left(\begin{bmatrix}9\\4\end{bmatrix}\right)=\begin{bmatrix}13\\0\\0\\8\\3\end{bmatrix}$$

For each matrix, determine if its transformation is (a) one-to-one but not onto, (b) onto but not one-to-one, (c) both, or (d) neither. Justify your answer.

14.

$$\begin{bmatrix}1&2&1&3\\1&3&1&3\\1&5&1&4\end{bmatrix}$$

15.

$$\begin{bmatrix}1&1&2\\2&3&5\\-1&-3&-3\end{bmatrix}$$

16.

$$\begin{bmatrix}5&0&5\\0&0&0\\4&0&7\end{bmatrix}$$

17.

$$\begin{bmatrix}1&2\\0&1\\-2&-2\\1&3\\1&4\end{bmatrix}$$

18.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

19.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

20.

$$\begin{bmatrix} 1 & 2 & -1 & 6 & 9 \\ 2 & 5 & -5 & 13 & 22 \\ -3 & -4 & -3 & -16 & -19 \end{bmatrix}$$

21.

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

22.

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & -6 & 7 \\ 0 & 0 & -8 & 9 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

23.

$$\begin{bmatrix} 12 & 45 & -3 & 20 & 1 \\ 0 & 24 & 121 & 0 & -47 \\ 0 & 0 & 252 & 44 & 46 \\ 0 & 0 & 0 & 21 & -44 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix}$$

24.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -13 & 6 & -39 & 4 \\ 2 & 9 & 6 & 7 \\ 5 & 4 & 15 & 16 \\ 0 & 1 & 0 & 65 \\ 1 & 1 & 1 & 1 \\ -12 & 3 & -36 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

25.

Draw the image of the unit square under the following linear transformations.

26.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

27.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

28. If $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto, then A is a square matrix.

29. If $\mathbf{x} \mapsto A\mathbf{x}$ is neither one-to-one or onto, and the reduced echelon form of A only has 0s and 1s, then one of the columns of A is $\mathbf{0}$.

30. For any matrix $A \in \mathbb{R}^{m \times n}$ where $m > n$, it is not possible for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one.

31. For any matrix A, if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then it is also one-to-one.

32. If A is an $m \times n$ matrix and $m < n$, then the range of $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

33. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $T(\mathbf{v}) = A\mathbf{v}$ for all vectors $\mathbf{v}$ in the domain of T, then A is an $m \times n$ matrix.

34. The matrix that implements a linear transformation is unique.

More Difficult Problems

35. Consider the following linear transformation T.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_3 \\ 2x_3 \end{bmatrix}$$

Determine a set of linearly independent vectors which span the range of T.

36. Determine the matrix which implements the linear tranformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the x_1x_2-plane.

37. Determine the matrix which implements the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ that repeats the input vector, e.g.,

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}$$

38. Determine the matrix which implements the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the line $y = x$.

39. Determine the matrix which implements the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the line $y = \tan(\frac{3\pi}{8})x$.

40. Determine the matrix which implements the transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ that swaps the first and second entries of its input, e.g.,

$$T\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right) = \begin{bmatrix}2\\1\\3\\4\end{bmatrix}$$

41. Consider the following transformation.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \mapsto \begin{bmatrix}(x^3 + y^3 + z^3)^{1/3}\\ y + z\end{bmatrix}$$

   (a) Demonstrate that the above transformation is homogeneous.

   (b) Determine if the above transformation is not linear. Justify your answer.

42. Show that the transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \mapsto \begin{bmatrix}\min(v_1, 100)\\ \min(v_2, 100)\\ \min(v_3, 100)\\ \min(v_4, 100)\end{bmatrix}$$

   is not linear.

43. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation which reflects vectors across the xy plane (i.e., across the plane given by the linear equation $z = 0$) and that $S : \mathbb{R}^3 \to \mathbb{R}^3$ the transformation which rotates vectors around $\text{span}\{[1\ \ 1\ \ 0]^T\}$ by 180 degrees. Determine the matrix which implements $S \circ T$, the composition of S and T (recall that $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$).

44. Considering the transformation T implemented by the following matrix.

$$\begin{bmatrix} \cos 2 & 0 & -\sin 2 \\ 0 & 1 & 0 \\ \sin 2 & 0 & \cos 2 \end{bmatrix}$$

Describe geometrically what T does. Then find a vector $\mathbf{v}$ whose span is not changed by this transformation (i.e., $\text{span}\{\mathbf{v}\} = \text{span}\{T(\mathbf{v})\}$).

45. Consider the following linear transformations.

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_2 \\ x_1 - x_2 \end{bmatrix} \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ -x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

(a) Determine the matrix that implements $S \circ T$, the composition of S and T. That is, determine the matrix A such that the transformation $\mathbf{x} \mapsto S(T(\mathbf{x}))$ is equivalent to the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

(b) Determine a general form solution that describes the preimages of the following vector $\mathbf{b}$ under the linear transformation $S \circ T$. That is determine a general form solution where $S(T(\mathbf{x})) = \mathbf{b}$ if and only if $\mathbf{x}$ is in the solution set described by your general form solution.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 8 \\ -3 \end{bmatrix}$$

Challenge Problems

46. Determine the matrices which implement the linear transformations that rotates vectors in $\mathbb{R}^3$ 120 degrees about

$$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

You should determine *two* matrices, one for clockwise rotation and the other for counterclockwise rotation.

47. Determine the matrices which implement the linear transformations that reflects vectors across the plane defined by the linear equation $x + y + z = 0$, and then rotates vectors 60 degrees about

$$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

You should determine *two* matrices, one for clockwise rotation and the other for counterclockwise rotation.

48. Determine the matrix which implements the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the plane defined by the linear equation $x + y + z = 0$.

49. Consider the following $\mathbb{R}^3$ rotation matrices.

$$A = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 180^\circ & -\sin 180^\circ \\ 0 & \sin 180^\circ & \cos 180^\circ \end{bmatrix}$$

The matrix A rotates vectors around the x_3-axis by 45 degrees, and B rotates vectors around the x_1-axis by 180 degrees.

(a) Determine A^{-1}.

(b) Calculate ABA^{-1}.

(c) Describe what the transformation implemented by ABA^{-1} does geometrically.

Chapter 6

Matrix Algebra and Invertibility

Basic Exercises

$$A = \begin{bmatrix} 6 & 1 \\ -1 & -3 \\ 8 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & -5 \\ -6 & -2 \\ -3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} -4 & 0 \\ -8 & 5 \\ -4 & -1 \end{bmatrix}$$

Compute the following expressions.

1. $A + B$
2. $2A - 4B - 3C$
3. $A^T A + B^T B$
4. $AC^T + CA^T$

Compute the following matrix multiplications, if possible. If it is not possible, explain why.

5.

$$\begin{bmatrix} -2 & 6 \\ -2 & -7 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -4 & 8 & 4 \\ -10 & -4 & 9 \end{bmatrix}$$

6.

$$\begin{bmatrix} 8 & 6 & -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \\ 5 \end{bmatrix}$$

7.

$$\begin{bmatrix} -3 \\ 6 \\ 8 \\ 5 \end{bmatrix} \begin{bmatrix} -9 & -7 & 0 & -8 \end{bmatrix}$$

8.

$$\begin{bmatrix} 7 & -10 \\ -6 & 6 \\ 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -8 & 9 \\ -10 & 0 \\ -9 & 1 \\ -4 & -3 \end{bmatrix}$$

9.

$$\begin{bmatrix} 2 & 3 & -1 \\ 7 & 4 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

12.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

13.

$$\begin{bmatrix} 1 & -1 & 4 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

14.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 & 2 \end{bmatrix}$$

15.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

16.

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$ are vectors in $\mathbb{R}^n$.

17.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{2023}$$

18.

$$\begin{bmatrix} \cos 2 & -\sin 2 & 0 \\ \sin 2 & \cos 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{2023}$$

Let A be matrix such that A^{-1} is defined as below. Use this to determine the solution to the matrix equations of the form $A\mathbf{x} = \mathbf{b}_i$, where each $\mathbf{b}_i$ is defined below.

19.

$$A^{-1} = \begin{bmatrix} -10 & -10 & -1 \\ -2 & -4 & -3 \\ -8 & -6 & 1 \end{bmatrix} \qquad \mathbf{b}_1 = \begin{bmatrix} -5 \\ 6 \\ -1 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} -1 \\ -3 \\ -10 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix}$$

Determine the inverse of the following matrices. If the matrix is not invertible, then explain why.

20.

$$\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$$

21.

$$\begin{bmatrix} -8 & -2 \\ -2 & -3 \end{bmatrix}$$

22.

$$\begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix}$$

23.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

24.

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

25.

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

26.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ -2 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \end{bmatrix}$$

27.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

28.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta + \sin\theta & \cos\theta - \sin\theta \end{bmatrix}$$

Simplify your solution as much as possible.

Determine the matrix in $\mathbb{R}^{4\times 4}$ that implements the following row operations, in order from top to bottom. That is, determine a matrix A such that AB is the result of applying the following row operations, from top to bottom, to B.

29.

$$\begin{aligned}
&R_2 \leftarrow R_2 - R_3 \\
&R_1 \leftrightarrow R_4 \\
&R_1 \leftarrow R_1 + 3R_2 \\
&R_3 \leftarrow R_3 - 5R_4 \\
&R_4 \leftarrow -2R_4
\end{aligned}$$

Determine the inverse of the following transformation, if it exists. Your solution should be in the form of a transformation, as given below.

30.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_3 \\ -3x_1 + x_2 - 4x_3 \\ x_1 + 2x_2 \end{bmatrix}$$

Suppose that A is a matrix in $\mathbb{R}^{3\times 3}$ such that the following sequence of row operations (from top to bottom) transforms A into the identity matrix. Determine the inverse of A. You should do this without determining A first.

31.

$$\begin{aligned}
&R_1 \leftrightarrow R_2 \\
&R_1 \leftarrow R_1 - R_2 \\
&R_3 \leftarrow R_3 - 3R_2 \\
&R_1 \leftarrow R_1 + 3R_2 \\
&R_3 \leftarrow R_3 - 5R_1 \\
&R_1 \leftrightarrow R_2
\end{aligned}$$

32.

$$\begin{aligned}
&R_2 \leftarrow R_2 - R_1 \\
&R_3 \leftarrow R_3 - R_1 \\
&R_3 \leftarrow R_3 - 4R_2 \\
&R_1 \leftarrow R_1 - R_3 \\
&R_1 \leftarrow R_1 + 2R_2
\end{aligned}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

33. For any two matrices A and B, if $A + B$ is defined then $A + B = B + A$.
34. For all matrices A and B in $\mathbb{R}^{m\times n}$ and C in $\mathbb{R}^{n\times m}$, we have $(A + B + C^T)^T = C + B^T + A^T$.

35. For any square matrices A and B in $\mathbb{R}^{n\times n}$, if $A = A^T$ and $B = B^T$ then $AB = (AB)^T$.

36. For all matrices A and B such that AB is defined, we have $AB \neq BA$.

37. For any matrix A, if A has n distinct eigenvalues, then it is invertible.

38. If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$ in the span of the columns of A, then A is invertible.

39. For any matrix A and vector $\mathbf{v}$, if $\mathbf{v}^T A$ is defined, then A is single column.

40. For any matrices A and B, if $A^{-1} = B^{-1}$, then $A = B$.

41. For any matrices A and B, if there is a unique matrix X such that $AX = B$, then A is invertible.

42. For any matrix $A \in \mathbb{R}^{n\times n}$ and diagonal matrix $D \in \mathbb{R}^{n\times n}$, it must be that $AD = DA$.

43. If A and B are invertible, then so is $A + B$.

44. If A and B are invertible, then so is BA.

45. If A and B in $\mathbb{R}^{n\times n}$ are invertible, then so is $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$, the matrix in $\mathbb{R}^{2n\times 2n}$ gotten by stacking copies of A and B.

46. For any matrices A and B, if there is a unique matrix X such that $AX = B$, then A is invertible.

47. For any matrices A and B, if $AB = 0$, then $A = 0$ or $B = 0$.

48. For any matrices $A, B \in \mathbb{R}^{n\times n}$, if $AB = I$, then $BA = I$.

49. For any matrices A and B, if $AB = BA$, then $A = B$.

50. If A and B are symmetric and $AB = BA$ then AB is symmetric.

51. For any matrix $A \in \mathbb{R}^{n\times n}$, if the columns of A^3 span all of $\mathbb{R}^n$, then the columns of A are linearly independent.

52. For any matrix $A \in \mathbb{R}^{n\times n}$, the matrix $A + A^T$ is symmetric.

53. If AB and BA are symmetric, then A and B are symmetric.

54. For any matrix $A \in \mathbb{R}^{n\times n}$, if AA^T is symmetric, then so is A.

55. If $A \in \mathbb{R}^{n\times n}$ has zeros along its diagonal, then A is not invertible.

56. If $A \in \mathbb{R}^{n\times n}$ has a row of all zeros, then A is not invertible.

57. If $A \in \mathbb{R}^{2\times 2}$ and A^{-1} has integer entries then the determinant of A is 1.

58. For any real numbers a and b, the matrix $\begin{bmatrix} a & a^2 \\ b & ab \end{bmatrix}$ is singular (i.e., not invertible).

59. For any square matrices A and B, if $AB = I$, then $AB = BA$.

60. The **Hadamard product** of two matrices is defined as

$$(A \circ B)_{ij} = A_{ij} B_{ij}$$

In other words, A and B are multiplied entry-wise. For any invertible matrices A and B, if $A \circ B$ is invertible, then $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$.

61. For any matrices A and B, if $AB = I$ then A is invertible and $B = A^{-1}$.

62. For any two symmetric matrices A and B, if AB is defined then $(AB)^T = A^T B^T$.

63. A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if A^T is invertible.

More Difficult Problems

64. Let A be as defined below. Is it possible to write the inverse of A as a power of A? If so, determine the smallest positive integer n such that $A^n = A^{-1}$.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

65. Determine the smallest positive integer n such that $A^n = A^{-1}$.

$$A = \begin{bmatrix} \cos\frac{\pi}{9} & -\sin\frac{\pi}{9} & 0 \\ \sin\frac{\pi}{9} & \cos\frac{\pi}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

66. Compute the following matrix expression. Your answer should be a single matrix with entries given in terms of n.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-n}$$

67. Suppose that A and B are invertible matrices such that $AB^T X A^{-1} B = I$ for some matrix X. Determine X in terms of A and B.

68. Let A, B, and C such that $A = A^{-1}$ and $C = C^T$ and

$$A(C^{-1}(AB)^T)^T C$$

is well-defined. Simplify this expression using the algebraic properties of matrix operations.

69. Determine a matrix in $\mathbb{R}^{2\times 2}$ with all nonzero entries that is equal to its inverse. *Hint.* Use the closed-form equation for the inverse of a 2×2 matrix.

70. Determine a matrix B with integer entries such that the following equality holds.

$$\begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & 2 \end{bmatrix} B = I$$

71. Explain why it is not possible to determine a matrix B such that the following equality holds.

$$B \begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & 2 \end{bmatrix} = I$$

72. Consider the following matrix and vector.[1]

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Compute the following matrix-vector multiplications.

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \left(A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

(b) Determine a general form solution for the solution set of the matrix equation $A\mathbf{x} = \mathbf{b}$.

(c) Use your solution to the previous part to find a **nonzero** vector $[v_1 \ \ v_2 \ \ v_3]^T$ such that

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \left(A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = 0$$

73. Suppose that A is a matrix such that

$$A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\det(A) = 1$$

Determine the inverse of A.

74. Determine a matrix A and two values k and h such that

$$A \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 0 \end{bmatrix}$$

[1] Credit to Vishesh Jain for suggesting a version of this problem.

75. Repeat the previous problem with

$$A\begin{bmatrix} 1 & -1 \\ 3 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & -1 \\ 3 & -2 \\ -1 & 3 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 0 \end{bmatrix}$$

Challenge Problems

76. Determine *three* matrices in $\mathbb{R}^{2\times 2}$ that satisfy the following equation. In particular, you must demonstrate that each matrix in your solution satisfies the equation.

$$X^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

77. Determine the reduced echelon form of the matrix

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

in terms of a, b, c, d. Show your work.

78. Determine two invertible matrices A and B such that $AB^{-1} = -BA^{-1}$.

79. Let A and B be two invertible matrices in $\mathbb{R}^{n\times n}$ such that $AB^{-1} = -BA^{-1}$. Determine the inverse of the following matrix in $\mathbb{R}^{2n\times 2n}$.

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Chapter 7

Elementary Matrices and LU Factorization

Basic Exercises

Determine an LU factorization of the following matrices.

1.
$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

2.
$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$$

3.
$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 3 & 2 & 10 & 9 \\ -3 & 2 & -2 & -9 \end{bmatrix}$$

4.
$$\begin{bmatrix} 1 & -1 & 4 & 0 & -5 \\ -2 & 3 & -13 & -1 & 8 \\ 2 & -4 & 18 & 3 & -2 \\ -3 & 5 & -22 & -4 & 3 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

5. Let $E \in \mathbb{R}^{n\times n}$ denote the elementary matrix that implements $R_1 \leftarrow 3R_1$. For any matrix $A \in \mathbb{R}^{n\times n}$, if $\det(EA) = 1$ then $\det A^{-1} = \frac{1}{3}$.

6. For any matrix $A \in \mathbb{R}^{m\times n}$, if $A = LU$ where L and U form an LU-factorization of A, then L is invertible.

7. For any matrix $A \in \mathbb{R}^{n\times n}$, if $A = LU$ where L and U form an LU-factorization of A, and if L is invertible, then so is A.

More Difficult Problems

8. (a) Determine the 3×3 matrix E which implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_1, R_2) \\ &R_1 \leftarrow 3R_1 \\ &R_3 \leftarrow R_3 + 2R_2 \end{aligned}$$

(b) Determine values for i through m such that E^T implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_i, R_j) \\ &R_k \leftarrow 3R_k \\ &R_l \leftarrow R_l + 2R_m \end{aligned}$$

(c) Using the previous part, compute AE where

$$A = \begin{bmatrix} 11 & 22 & 33 \\ 11 & 22 & 33 \\ 11 & 22 & 33 \end{bmatrix}$$

9. Suppose that LU is the LU factorization of a matrix A, and let B be a matrix such that $BU = I$. Determine the inverse of A in terms of L, U, and B.

10. Let A be a 5×2026 matrix such that $\operatorname{rank} A = 4$, which has an LU decomposition where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & 1 \end{bmatrix}$$

Determine if $\mathbf{v}$ in $\operatorname{Col} A$, where

$$\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ -11 \\ 5 \\ -12 \end{bmatrix}$$

Chapter 8

Linear Dynamical Systems and Markov Chains

Basic Exercises

For each of the following stochastic matrices A, answer the following items:

- Draw the state diagram for the corresponding Markov chain. You should label each node with a positive integer for the corresponding column of A (e.g, the node corresponding to the first column should be labeled "1", to the second column labeled "2", and so on).
- Determine if A is regular, and write down the smallest k such that A^k has strictly positive values. If it is not regular, justify your answer.
- Determine the general form solution of the equation $(A - I)\mathbf{x} = \mathbf{0}$. You may use a computer to do this, but you should express your answer in fractions, not decimals.
- Determine a steady state vector for A. If the steady state vector is unique, note this. You must do this by hand and show your work.

1.

$$\begin{bmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{bmatrix}$$

2.

$$\begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.4 & 0.3 & 0 \\ 0 & 0.7 & 0.9 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

4.

$$\begin{bmatrix} 0.9 & 0.9 \\ 0.1 & 0.1 \end{bmatrix}$$

5.

$$\begin{bmatrix} 0.2 & 0 & 0 \\ 0.4 & 1 & 0.3 \\ 0.4 & 0 & 0.7 \end{bmatrix}$$

6.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

7.

$$\begin{bmatrix} 0.2 & 0 & 0.3 \\ 0.8 & 0.5 & 0 \\ 0 & 0.5 & 0.7 \end{bmatrix}$$

8.

$$\begin{bmatrix} 0 & 0 & 0.3 \\ 1 & 0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$

9.

$$\begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & 0.3 & 0.4 \\ 0 & 0.7 & 0.1 \\ 0 & 0 & 0.5 \end{bmatrix}$$

11.

$$\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

12.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

13.

$$\begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

14.

$$\begin{bmatrix} 0.4 & 0.8 & 0.5 & 0 \\ 0.6 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

15. For any matrix $A \in \mathbb{R}^{n\times n}$, if A is regular and stochastic then $\dim \text{Nul}(A - I) = 1$.
16. If A is a stochastic matrix, then the all ones vector $\mathbf{1}$ is an eigenvector for A^T.
17. If a stochastic matrix has a unique steady state vector, then the corresponding Markov chain converges to it regardless of starting state.
18. A stochastic regular matrix $A \in \mathbb{R}^{n\times n}$ has a smallest integer k such that A^k has strictly positive entries. For such a matrix, $k \le n$.
19. Every stochastic matrix has at least one steady state vector.
20. If A and B are stochastic matrices, then so is AB.
21. Every A stochastic matrix is invertible.
22. For any stochastic matrix A, if A has a unique stationary state, then it is regular.
23. If A is a square matrix with strictly positive entries, then there is diagonal matrix D such that AD is stochastic.

More Difficult Problems

24. Suppose you've been watching the stock price of your favorite company and you've discovered the following trends based on the data you've taken:

    - If the price is **STEADY** there's a 70% chance it will remain steady the next day, but a 20% chance it will go **HIGH** and a 10% chance it will dip **LOW**.
    - If the price is **HIGH** there's a 55% chance it will remain high the next day, but a 40% chance it will go back to being **STEADY**, and just a 5% chance it will dip all the way down to **LOW**.

- If the price is **LOW**, there is a 60% change it will remain low the next day, but a 30% chance it will reset to **STEADY**, and a 10% chance it will suddenly jump to **HIGH**.

Suppose that the price is currently **STEADY**. In the long term, is more likely to remain **STEADY**, go up **HIGH**, or dip down **LOW**? Justify your answer.

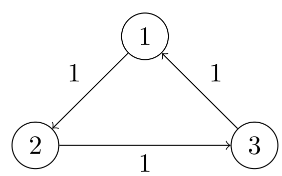
25. Suppose that we're trying to predict the preformance of the A_5 Soccer Club based on statistics that we've gathered on their recent games. We've found that:

    - After a win, they have a 70% chance of winning, a 15% chance of drawing, and a 15% chance losing their next game.
    - After a draw, they have a 20% chance of winning, a 50% chance of drawing, and a 30% chance of losing their next game.
    - After a loss, they have a 20% chance of winning, a 70% chance of drawing, and a 10% chance of losing their next game.

    (a) If they win their first game, how should we expected them to perform overall? That is, what is the expected percentage of wins, draws, and losses in the long term?

    (b) If they lose their first game, what will their overall record tend towards?

26. Consider the following state diagram.



If an edge between two states is not present, the probability of transitioning in a single step is 0. For example, the probability of transitioning from state 3 to state 2 in a single step is 0.

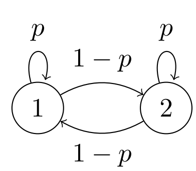
    (a) Determine the transition matrix T for the above diagram. You should write T such that the ith column of T corresponds to the transitions from state i. (In the following parts, T refers to the matrix you determined in this part.)

    (b) Compute T^2, T^3 and T^{2023}.

    (c) Determine if T is regular. Justify your answer.

    (d) Determine if T have a unique steady-state distribution. Justify your answer.

27. Consider the following state diagram.



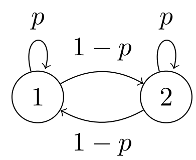
    (a) Determine the transition matrix T for the above diagram in terms of p. In the following parts, T will refer to this matrix.

    (b) Given $0 < p < 1$, determine $\lim_{k\to\infty} T^k \mathbf{e}_1$. Justify your answer.

Challenge Problems

28. Demonstrate that any invertible stochastic matrix A with a stochastic inverse must be a permutation matrix. That is, every column and every row of A has a single 1, with the rest of the entries being 0s. *Hint:* Consider what can be inferred about the entries of a stochastic A^{-1} from its action on $A\mathbf{e}_i$ for any elementary basis vector $\mathbf{e}_i$.

29. Let T denote the transition matrix of the following state diagram.



(a) Determine λ_2, the *second* largest eigenvalue of T, and a corresponding eigenvector. *Hint:* T is symmetric and, hence, orthogonally diagonalizable.[1]

(b) Give a closed-form solution for $T^k[(1-q) \quad q]^T$ in terms of p, q and k.

[1] The second largest eigenvalue tells us about the *rate of convergence* to the steady-state distribution. The smaller $|\lambda_2|$ is, the faster the convergence to the steady-state distribution.

Chapter 9

Vector Spaces and Subspaces

Basic Exercises

For each of the following exercises, justify your answer.

1. Given $A \in \mathbb{R}^{3\times 6}$, determine value of n such that $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

2. Given $A \in \mathbb{R}^{10\times 13}$, determine the minimum dimension of $\operatorname{Nul} A$.

3. Given $A \in \mathbb{R}^{7\times 5}$ and $\operatorname{rank} A = 4$, determine $\dim(\operatorname{Nul} A)$.

4. Determine if v is in $\operatorname{Nul} A$ where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 6 & 0 \\ 5 & 7 & 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

5. Determine if $\mathbf{v}$ is in $\operatorname{Col} A$, where $\mathbf{v}$ and A are defined as in the previous exercise.

6. Without performing any row operations, determine $\operatorname{rank} A$ where

$$A = \begin{bmatrix} 2 & 1 & -8 & 3 \\ -1 & 3 & 4 & 2 \\ 3 & 2 & -12 & 5 \\ 1 & -2 & -4 & -1 \end{bmatrix}$$

For each of the following matrices, determine a basis for its column space and a basis for its null space.

7.

$$\begin{bmatrix} 1 & -5 \\ -2 & 10 \end{bmatrix}$$

8.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9.

$$\begin{bmatrix} 0 & -2 & 2 \\ -2 & 3 & -9 \\ -1 & -2 & -1 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & -4 & 3 & -3 \\ -2 & 8 & -6 & 7 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & -4 & -3 \\ -3 & 12 & 10 \\ -2 & 8 & 8 \\ -1 & 4 & 2 \end{bmatrix}$$

12. The matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5 \quad \mathbf{a}_6 \quad \mathbf{a}_7 \quad \mathbf{a}_8]$ which is row-equivalent to the following matrix.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 5 & 0 & -7 \\ 0 & 1 & 14 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

13.

```
import sympy
sympy.Matrix(
  [[1,   2,   4,   3,  -4,  1],
   [-3, -4, -10,  -8,  13, -3],
   [ 5,  6,  16,  10, -13,  9],
   [-7, -8, -22, -12,  13, -9],
   [13, 18,  44,  32, -47, 11]]
)
```

For each of the following subspaces, determine a basis.

14.

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -8 \end{bmatrix} \right\}$$

15.

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 18 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix} \right\}$$

Determine the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ where $\mathbf{u}$ and $\mathcal{B}$ are defined below.

16.

$$\mathbf{u} = \begin{bmatrix} 8 \\ -12 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$$

17.

$$\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

18.

$$\mathbf{u} = \begin{bmatrix} -17 \\ 15 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$$

19.

$$\mathbf{u} = \begin{bmatrix} 5 \\ 29 \\ -80 \\ 42 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} -11 \\ -19 \\ 17 \\ 9 \end{bmatrix}, \begin{bmatrix} 9 \\ 17 \\ -20 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -18 \\ 18 \end{bmatrix} \right\}$$

Determine the change-of-basis matrix for the following bases.

20.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \end{bmatrix} \right\}$$

21.

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 10 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \\ 3 \end{bmatrix} \right\}$$

For each of the following matrices, determine a vector that is *not* in its column space.

22.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

23.

$$\begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & -1 \\ 1 & 2 & 9 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

24. There is a unique basis for any subspace.

25. A 7×4 matrix A may have $\dim(\operatorname{Nul} A) = 5$.

26. A 3×6 matrix A may have $\dim(\operatorname{Nul} A) = 2$.

27. For any matrix $A \in \mathbb{R}^{n \times n}$, if A is invertible then $\operatorname{rank} A = n$.

28. For any matrix $A \in \mathbb{R}^{m \times n}$, $\operatorname{Col}(A)$ is the same as the set of vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.

29. For any matrix A, we have $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

30. A basis is a spanning set that is as large as possible.

31. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps $\mathbb{R}^3$ to a plane has a trivial kernel (i.e., if $T(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$).

32. If $\mathcal{B}$ is the standard basis for R^n, then for any $\mathbf{x} \in \mathbb{R}^n$ we have that $[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$

33. For any matrix A, the dimension of the null space of A is at most the rank of A.

More Difficult Problems

34. For a matrix $A \in \mathbb{R}^{m \times n}$, consider the set of vectors $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{e}_1\}$ (*Recall:* $\mathbf{e}_1$ is the first standard basis vector).

    (a) Determine if this set if closed under addition. Justify your answer.

    (b) Determine if this set is closed under scaling. Justify your answer.

    (c) Determine if this set is a subspace of $\mathbb{R}^n$. Justify your answer.

35.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

List all possible subsets of the above vectors that form a basis of the subspace $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.[1]

36. Consider the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

List all possible subsets of the above vectors that form a basis of the subspace $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

[1] The process we teach for determining a basis gives one choice, but there may be multiple choices that are suitable.

37. Consider the 3-dimensional vector space of all quadratic polynomials $Q = \{ax^2+bx+c | a, b, c \in \mathbb{R}\}$ and consider the linear derivative map $\frac{d}{dx} : Q \to Q$ defined by $\frac{d}{dx}(ax^2 + bx + c) = 0x^2 + 2ax + b$.

   (a) Using the standard basis given by $\{x^2, x, 1\}$, determine a 3×3 matrix A that implements $\frac{d}{dx}$.

   (b) Determine a basis for $\operatorname{Col} A$.

   (c) Determine basis for $\operatorname{Nul} A$.

   (d) Determine $\operatorname{rank} A$ and $\dim(\operatorname{Nul} A)$.

38. Let A be an $m \times n$ matrix and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Let $\mathbf{b}$ denote the vector $A\mathbf{v}$.

   (a) Show that if $\mathbf{w} \in \operatorname{Nul} A$, then $\mathbf{v} + \mathbf{w}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

   (b) Show that if $\mathbf{b} \neq 0$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is not a subspace of $\mathbb{R}^n$.

   (c) A set H is an **affine** subspace of $\mathbb{R}^n$ if there is a subspace U of $\mathbb{R}^n$ and a vector $\mathbf{o}$ such that

$$H = \{\mathbf{u} + \mathbf{o} \mid \mathbf{u} \in U\}$$

   Show that the solution set of $A\mathbf{x} = \mathbf{b}$ is an affine subspace of $\mathbb{R}^n$ if it is nonempty. In particular, choose a vector $\mathbf{o}$ and subspace U.

39. Consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   (a) Determine a basis for $\operatorname{Nul} A$.

   (b) Determine a linear equation over the variables x_1, x_2, x_3, x_4 whose solution set is $\operatorname{Col} A$ (that is, (x_1, x_2, x_3, x_4) satisfies the linear equation if and only if the vector $[x_1\ x_2\ x_3\ x_4]^T$ is in the column space of A).

40. Determine the linear equation whose solution set is the column space of the following matrix.

$$\begin{bmatrix} 1 & -4 & -10 \\ -4 & 17 & 42 \\ 1 & -2 & -6 \end{bmatrix}$$

41. Let A be a 4×2024 matrix where $\operatorname{rank} A = 3$. Further suppose that the LU-decomposition of A has

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 7 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   Determine the linear equation whose solution set is $\operatorname{Col} A$.

42. Consider the following two matrices.

$$A = \begin{bmatrix} 6 & -6 & -1 & 17 \\ 9 & -2 & 2 & 36 \\ -10 & -13 & 8 & -45 \\ -9 & 4 & -1 & -33 \\ 2 & 4 & 2 & 14 \end{bmatrix} \qquad B = \begin{bmatrix} -8 & 5 & -16 & 1 \\ 2 & 11 & 4 & 16 \\ 3 & -2 & 6 & 3 \\ 2 & -10 & 4 & 0 \\ 8 & 4 & 16 & 9 \end{bmatrix}$$

Also consider the following set of vectors.

$$H = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in \operatorname{Col} A \text{ and } \mathbf{v} \in \operatorname{Col} B\}$$

That is, H consists of all sums of pairs of vectors, one from $\operatorname{Col} A$ and one from $\operatorname{Col} B$.

(a) Show that H is a subspace of $\mathbb{R}^5$.

(b) Determine the dimension of H. Justify your answer. You may use a computer but you must determine any matrix you reduce along with its reduced echelon form.

Challenge Problems

43. The row space of a matrix $A \in \mathbb{R}^{m \times n}$ is the span of the rows of A, denoted $\operatorname{Row} A$. Show that $\dim(\operatorname{Row} A) + \dim(\operatorname{Nul} A) = n$.

44. In the vector space of all real-valued functions, find a basis for the subspaced spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

Chapter 10

Eigenvalues and Eigenvectors

Basic Exercises

Determine if $\mathbf{v}$ is an eigenvector of A. If it is, find its corresponding eigenvalue.

1.

$$A = \begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} -10 & -3 & -5 \\ 5 & -5 & -3 \\ 5 & 7 & -7 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$$

Determine if λ is an eigenvalue of A. If it is, find a basis for the corresponding eigenspace.

3.

$$A = \begin{bmatrix} -2 & 2 \\ -1 & -4 \end{bmatrix} \qquad \lambda = -3$$

4.

$$A = \begin{bmatrix} 5 & -6 & 2 \\ 1 & -2 & 2 \\ -1 & 6 & 2 \end{bmatrix} \qquad \lambda = 4$$

5.

$$A = \begin{bmatrix} 4 & -42 \\ 1 & -9 \end{bmatrix} \qquad \lambda = -3$$

6.

$$A = \begin{bmatrix} 3 & -8 & 8 \\ 8 & -13 & 8 \\ 2 & -2 & -3 \end{bmatrix} \qquad \lambda = -5$$

7.

$$A = \begin{bmatrix} 3 & -8 & 8 \\ 8 & -13 & 8 \\ 2 & -2 & -3 \end{bmatrix} \qquad \lambda = 4$$

For the following matrices, determine all eigenvalues and bases for the corresponding eigenspaces.

8.

$$\begin{bmatrix} -3 & 2 \\ -10 & 6 \end{bmatrix}$$

9.

$$\begin{bmatrix} 1 & 16 & -12 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

10.

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

Calculate the determinant of the following matrices.

11.

$$\begin{bmatrix} -1 & 1 & -1 \\ 3 & 3 & 3 \\ 1 & -3 & -2 \end{bmatrix}$$

12.

$$\begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

For each of the following matrices, calculate the determinant of its inverse.

13.

$$\begin{bmatrix} 1 & 3 & 4 \\ -5 & -4 & -3 \\ 2 & 0 & -5 \end{bmatrix}$$

For each of the following matrices, determine its characteristic polynomial. Use the characteristic polynomial to calculate its determinant.

14.

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

15.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 & 0 \\ 10 & -15 & 3 & 4 & 0 \\ -1 & 5 & 2 & 5 & 5 \end{bmatrix}$$

16.

$$\begin{bmatrix} 1 & 0 & 2 & 10 & 5 \\ 0 & 0 & 5 & -3 & 15 \\ 0 & 0 & 16 & 6 & -1 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 7 & 4 \end{bmatrix}$$

17.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

18. Every eigenspace of A is a null space (potentially of some other matrix).

19. Every matrix has at least one eigenvector.

20. Every matrix $A \in \mathbb{R}^{n\times n}$ has an eigenbasis, i.e., a set of eigenvectors that form a basis for $\mathbb{R}^n$.

21. Only square matrices can have eigenvectors.

22. If 0 is an eigenvalue of A, then A is not invertible.

23. The determinant of a matrix does not change under elementary row operations.

24. If $\det(A^2) = 1$, then $\det(A) = 1$.

25. A matrix with characteristic polynomial $(x + 2)^3$ has a single eigenspace with dimension 3.

26. For any square matrices A and B, if $A \sim B$ then $\det(A) = \det(B)$.

27. $\det(5A) = 5 \det A$ for any square matrix A.

28. $\det(A^T A) \geq 0$ for any square matrix A.

29. $\det(A + B) = \det(A) + \det(B)$ for any square matrices A and B.

30. If A is invertible, then $\det(ABA^{-1}) = \det(B)$.

31. For any matrix A, if A is square and $\det(A) = 0$, then the columns of A are linearly dependent.

32. For any matrix A and vector $\mathbf{v}$, if $\mathbf{v}$ is an eigenvector of the matrix A then it is also an eigenvector of the matrix A^2.

33. For any square matrices A and B, we have $\det(AB^T) = \det(A)\det(B)$

More Difficult Problems

34. Let A be the following matrix.

$$\begin{bmatrix} -17 & 28 & 14 \\ -7 & 11 & 7 \\ -7 & 14 & 4 \end{bmatrix}$$

(a) Determine if the following vectors are eigenvectors of A. For the ones that are, find their associated eigenvalues.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(b) Show that -3 is an eigenvalue of A without doing any row operations. *Hint:* Use the invertible matrix theorem.

(c) Find a basis for the eigenspace of A corresponding to the eigenvalue -3.

35. Let A and B be square matrices such that $\det A = 3.5$ and $\det B = -2$. Compute the determinant of the matrix $B(AB)^{-1}(AB)^T A$.

36. Consider the following reduction sequence

$$\begin{aligned}
R_1 &\leftarrow R_1 + R_2 \\
\mathsf{swap}&(R_2, R_3) \\
R_3 &\leftarrow R_3 + 5R_4 \\
R_2 &\leftarrow -3R_2 \\
R_5 &\leftarrow R_5 - 10R_3 \\
R_5 &\leftarrow R_5/11 \\
\mathsf{swap}&(R_5, R_3) \\
\mathsf{swap}&(R_1, R_2) \\
R_4 &\leftarrow R_4 + R_1 \\
R_2 &\leftarrow 5R_2 \\
R_1 &\leftarrow -R_1
\end{aligned}$$

Suppose that $A \in \mathbb{R}^{5\times 5}$ reduces to U by this sequence of reductions, where U is in *reduced* echelon form. Given $\operatorname{rank} A = 5$, determine $\det A$.

37. Let A be as in the previous problem. Given $\operatorname{rank} A = 4$, determine $\det A$.

38. The characteristic polynomial of the following matrix is $-\lambda^3 + 5\lambda^2 - 6\lambda$.

$$\begin{bmatrix} 0 & 3 & -3 \\ -2 & 6 & -4 \\ -2 & 3 & -1 \end{bmatrix}$$

Find the eigenvalues and bases for the eigenspaces.

39. For each of the following statements, find a counterexample by giving explicit 2×2 matrices A and B which falsify the statement. Justify your answer.

   (a) $\det(A+B) = \det(A) + \det(B)$ for any matrices A and B in $\mathbb{R}^{n\times n}$.

   (b) For any square matrix A, if $\det(A - \lambda I) = (\lambda - 1)^2$, then $\dim(\text{Nul}(A - I)) = 2$.

   (c) For any matrices A and B in $\mathbb{R}^{n\times n}$, if $\det(A - \lambda I) = \det(B - \lambda I)$ (i.e., they have the same characteristic polynomial) then A is similar to B.

40. Let A be a 5×5 matrix with characteristic polynomial $(\lambda - 2)^2(\lambda + 1)^2\lambda$ and $\text{rank}(A - 2I) = \text{rank}(A + I) = 3$.

   (a) Determine rank A. Justify your answer.

   (b) Determine if A is diagonalizable. Justify your answer.

41. Suppose that $AP = PD$ for a square matrix A, diagonal matrix D, and arbitrary $m \times n$ matrix P. Show that the nonzero columns of P are eigenvectors of A and find their corresponding eigenvalues in terms of entries of D.

42. Consider the following pair of vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

   (a) Determine the matrix equations whose solution set consists exactly of those vectors which are orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (b) Determine a general-form solution for the matrix equation from the previous part.

   (c) Determine the dimension of the subspace of vectors orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

43. Show that the angle between two vectors does not depend on their length. In other words, show that for any pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the angle between them is equal to the angle between $a\mathbf{u}$ and $b\mathbf{v}$ for any nonzero real numbers a and b.

44. Suppose the eigenvalues of a 3×3 matrix A are $\lambda_1 = 2$, $\lambda_2 = 1/2$, and $\lambda_3 = 1/4$ with corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} = \\ 2 \\ -2 \end{bmatrix} 4$$

   Given a starting state $\mathbf{x}_0 = [8, -9, 6]^T$, give a closed form expression for the state after k iterations: $A^k\mathbf{x}_0$. Describe what happens as $k \to \infty$.

45. Recall that counterclockwise rotation by angle θ in the plane can be implemented by the matrix below.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   Calculate $\det(R_\theta)$. *Hint.* The value does not depend on θ.

46. Let R_θ be as in the previous problem.

    (a) Determine the characteristic polynomial of R_θ as a function of θ.

    (b) Determine the values of θ for which R_θ has eigenvalues. For those values, also determine bases for the corresponding eigenspaces.

47. Give two 2×2 matrices where every nonzero vector is an eigenvector.

48. Let A be a matrix such that

$$A\begin{bmatrix}1\\1\\-3\end{bmatrix} = \begin{bmatrix}-1\\-1\\3\end{bmatrix} \qquad A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad A\begin{bmatrix}2\\-2\\-9\end{bmatrix} = \begin{bmatrix}4\\-4\\-18\end{bmatrix}$$

Without determining A, determine the vector

$$A^5\begin{bmatrix}3\\0\\-11\end{bmatrix}$$

You may use a computer, but you must show your work and justify your answer.

49. Let A be the matrix from the previous part. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\-9\end{bmatrix} \right\}$$

Determine the matrix C such that $C[\mathbf{v}]_{\mathcal{B}} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

50. Determine if the following matrix has an eigenbasis. Justify your answer.

$$\begin{bmatrix}1 & 2 & -4 & 1\\0 & 1 & 4 & 1\\0 & 0 & 0 & 3\\0 & 0 & 0 & 2\end{bmatrix}$$

51. Consider the following matrix.

$$A = \begin{bmatrix}-3 & 0 & 0 & 0 & 0\\-5 & 2 & 0 & 0 & 0\\1 & 3 & -1 & 0 & 0\\2 & 10 & 6 & -4 & 1\\-7 & -3 & -5 & -6 & 3\end{bmatrix}$$

    (a) Determine the characteristic polynomial of A in fully-factored from.

    (b) Determine the rank of A (i.e., the dimension of $\operatorname{Col} A$) and the nullity of A (i.e, the dimension of $\operatorname{Nul} A$). Justify your answer.

    (c) Determine the rank of $A + I$. Justify your answer

(d) Consider the following matrix.

$$B = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 2 & 10 & 6 & -4 & 1 \\ -7 & -3 & -5 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use the Rank-Nullity theorem to determine $\operatorname{rank} B$, $\dim \operatorname{Nul} B$ and $\dim \operatorname{Nul} B^T$.

Challenge Problems

52. Show that a pair of eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with distinct eigenvalues λ_1 and λ_2 is linearly independent.

53. Show that the determinant formula from lecture is correct. *Hint:* Consider applying elementary row operations via left multiplication by elementary row matrices, and use the fact that

$$\det(E_k \dots E_2 E_1 A) = \det E_k \dots \det E_2 \det E_1 \det A$$

54. Consider the following matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Verify that the following vectors form an eigenbasis of A. Also determine the eigenvalues for each eigenvector.

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

(b) Write the vector $\mathbf{e}_1$ in terms of the eigenbasis you found. In other words, determine α_1 and α_2 such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$$

Hint: Calcuate $\mathbf{v}_1 - \mathbf{v}_2$.

(c) Write down a closed-form solution for the linear dynamical system determined by A with initial vector $\mathbf{e}_1$.

(d) If you look at the formula given by the second component of your closed-form solution from the previous part, this gives a *non-recursive* definition for Fibonacci numbers. Write down this formula and use it to calculate F_{20}, the 20th fibonacci number (where $F_0 = 0$ and $F_1 = 1$).

55. A Jordan block J is a square matrix in $\mathbb{R}^{n \times n}$ of the following form.

$$\begin{bmatrix} \alpha & 1 & & \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ & & & \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{R}$.

(a) Determine the characteristic polynomial (in terms of n) of J.

(b) Determine the dimension of the eigenspace of J for the eigenvalue α.

(c) Construct six matrices, each with the characteristic polynomial $(\lambda - 3)^3(\lambda + 2)^2$, that achieve all possible combinations of eigenspace dimensions for $\lambda = 3$ (1,2, or 3) and for $\lambda = 2$ (1 or 2).

Chapter 11

Eigenbases and Diagonalization

Basic Exercises

Determine a diagonalization of the following matrix, if it exists. You can leave the rightmost factor in the form P^{-1}, i.e., you don't have to compute the inverse of P.

1. $$\begin{bmatrix} 9 & 4 \\ -14 & -6 \end{bmatrix}$$

2. $$\begin{bmatrix} 2 & -6 \\ 2 & -5 \end{bmatrix}$$

3. $$\begin{bmatrix} -6 & -7 \\ 7 & 8 \end{bmatrix}$$

4. $$\begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & -1 \\ 6 & 4 & -5 \end{bmatrix}$$

5. $$\begin{bmatrix} -1 & -3 & -6 \\ -8 & -6 & -18 \\ 4 & 3 & 9 \end{bmatrix}$$

   Hint: The characteristic polynomial of the above matrix is $\lambda^3 - 2\lambda^2 - 3\lambda$.

6. $$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 3 & -3 & 2 & 0 \\ -3 & 18 & -6 & -1 \end{bmatrix}$$

7.

$$\begin{bmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.

$$\begin{bmatrix} 0 & 2 & 2 \\ 5 & -6 & -5 \\ -9 & 12 & 11 \end{bmatrix}$$

In addition, define P so that its entries are integers.

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

9. All square matrices are diagonalizable.
10. Similar matrices have the same eigenvalues.
11. Similar matrices have the same eigenvectors.
12. If a matrix does not have n distinct eigenvalues, then it is not diagonalizable.
13. If a matrix is diagonalizable, then it is invertible.
14. A diagonalization of a matrix A if it exists, is unique.

More Difficult Problems

15. Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Determine bases for every eigenspace of A. That is for each eigenvalue λ of A, find a basis for $\mathrm{Nul}(A - \lambda I)$.

(c) Determine if A is diagonalizable. If it is, provide a diagonalization. Otherwise, justify your answer.

16. Consider the following matrix.

$$A = \begin{bmatrix} 103 & -8 & -47 \\ 24 & 1 & -12 \\ 198 & -16 & -90 \end{bmatrix}$$

Determine a matrix B such that $B^2 = A$. You can use a computer to find eigenvectors or to reduce matrices, but you must otherwise show your work and justify your answer. *Hint:* A is diagonalizable.

17. Suppose A is a 2×2 matrices such that

$$A\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} \qquad A\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-16\\8\end{bmatrix}$$

(a) Determine a diagonalization of A.

(b) Determine a matrix B such that $B^3 = A$.

18.

19. Consider the following matrix.

$$A = \begin{bmatrix}-2 & 4\\-2 & 4\end{bmatrix}$$

(a) Determine a diagonalization of A.

(b) Using the same technique as in the previous problem determine the matrix 2^A.

Challenge Problems

20. Consider an arbitrary 2×2 matrix of the following form.

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

(a) Determine an expression for the characteristic polynomial of A in terms of a, b, c, and d.

(b) Recall that for a quadratic polynomial $p(x) = ix^2 + jx + k$, the discriminant $j^2 - 4ik$ tells us how many roots p has, i.e., p has 0, 1 or 2 roots if the discriminant is less than 0, equal to 0, or greater than 0, respectively. Use this to derive an expression E in terms of a, b, c, and d where A has 0, 1, or 2 eigenvalues if $E < 0$, $E = 0$, or $E > 0$, respectively.

(c) Using the expression from the previous part, argue that *every* 2×2 *matrix with positive entries has two distinct eigenvalues.* Note that this implies every 2×2 matrix with positive entries is diagonalizable. *Hint:* Try to write the expression from the previous part so that it is of the form '$(\square - \square)^2 + 4\square\square$' and reason about why this must be positive.

21. Consider the following matrices

$$A = \begin{bmatrix}4 & 1\\-2 & 5\end{bmatrix} \qquad B = \begin{bmatrix}4 & -6\\8 & -8\end{bmatrix}$$

Determine the characteristic polynomial of $2^A 2^B$.

Chapter 12

Analytic Geometry and Orthogonality

Basic Exercises

For each pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, determine:

- the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$;
- the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$;
- the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$;
- the unit length normalizations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

You must simplify the expression as much as you can before giving the approximate result to a couple decimal places.

1.
$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

2.
$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

3.
$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 5 \\ 7 \\ 4 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$$

   Additionally, without doing any further calculations determine approximately the angle between $\mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. Justify your answer.

Determine if each of the following set of vectors is an orthogonal set.

4.

$$\left\{ \begin{bmatrix} 9 \\ -18 \\ -18 \end{bmatrix}, \begin{bmatrix} -2 \\ -14 \\ 13 \end{bmatrix} \right\}$$

5.

$$\left\{ \begin{bmatrix} 0 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -6 \end{bmatrix} \right\}$$

Express the vector $\mathbf{v}$ in terms of the given orthonormal basis $\mathcal{B}$.

6.

$$\mathbf{v} = \begin{bmatrix} -8 \\ 9 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

7.

$$\mathbf{v} = \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} \right\}$$

Determine the projection of $\mathbf{v}$ onto the span of the given set of vectors.

8.

$$\mathbf{v} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \qquad U = \left\{ \begin{bmatrix} 7 \\ -9 \end{bmatrix} \right\}$$

9.

$$\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \qquad U = \left\{ \begin{bmatrix} -2 \\ 0 \\ -10 \end{bmatrix} \right\}$$

10.

$$\mathbf{v} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} \qquad U = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \right\}$$

Determine an orthogonal diagonalization of the following matrices.

11.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

12. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

13. For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A$.

14. There can be a linear dependence relationship between vectors in an orthogonal set.

15. For any $A \in \mathbb{R}^{m \times n}$, there is an eigenbasis of $\mathbb{R}^n$ for $A^T A$.

16. In $\mathbb{R}^4$, any set of vectors that has 5 members cannot be an orthogonal set.

17. Orthogonal matrices are invertible.

18. Orthogonal matrices have determinant 1.

19. For any matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{v} \in \mathbb{R}^n$, if $\|A\mathbf{v}\| = \mathbf{0}$ then $\mathbf{v} \in \operatorname{Nul} A$.

20. For an $m \times n$ matrix with orthogonal columns, it may be the case that $m < n$.

21. For any matrix $U \in \mathbb{R}^{n \times n}$, if U is orthogonal then $\det U = \pm 1$.

22. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

23. For any vector $\mathbf{y} \in \mathbb{R}^n$ and any subspace W of $\mathbb{R}^n$, the vector $\mathbf{y} - \operatorname{proj}_W \mathbf{y}$ is orthogonal to W.

24. For any matrix $A \in \mathbb{R}^{n \times n}$, if A has an eigenbasis of $\mathbb{R}^n$ then A is symmetric.

25. For any matrix A with orthogonal columns, $A^T A = I$.

26. Every orthogonal set is linearly independent.

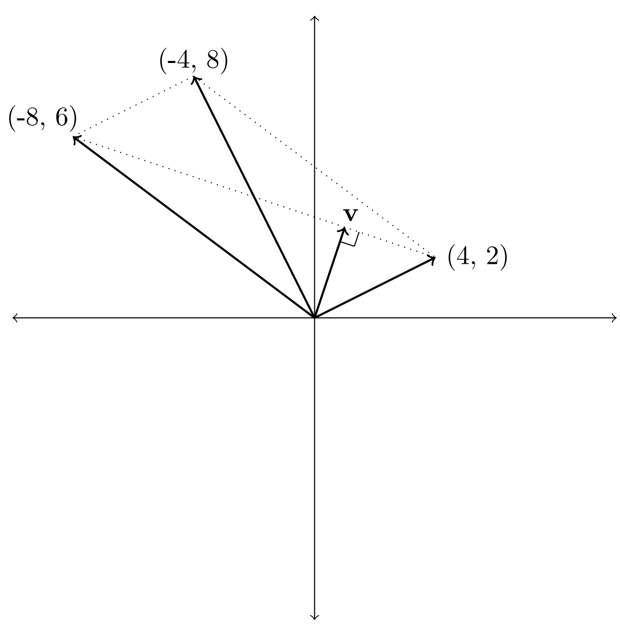
27. For any matrix A, if A is orthogonally diagonalizable, then so is A^T.

28. $\|A\mathbf{v}\| = \|A^2\mathbf{v}\|$ for any orthogonal matrix $A \in \mathbb{R}^{n \times n}$ and any vector $\mathbf{v} \in \mathbb{R}^n$.

29. For any vector $\mathbf{u} \in \mathbb{R}^n$, if $\|\mathbf{u}\| = 1$ then the matrix $\mathbf{u}\mathbf{u}^T$ implements projection onto $\operatorname{span}\{\mathbf{u}\}$.

More Difficult Problems

30.



(a) Determine the distance between $(-8, 6)$ and $(4, 2)$.

(b) Determine the length of the vector $\mathbf{v}$, whose endpoint is on the line segment between $(-8, 6)$ and $(4, 2)$ and which forms a $90°$ angle with that line segment.

(c) Use these values to determine the area of the parallelogram formed by $(0, 0)$, $(4, 2)$, $(-4, 8)$, and $(-8, 6)$.

(d) Determine a matrix in $\mathbb{R}^{2\times 2}$ that transforms the unit square into the parallelogram from the previous part. Recall that the unit square is the set of points in

$$\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$

(e) Calulate the determinant of the matrix from the previous part.

31. Determine if the following six vectors form an orthogonal set. Justify your answer. *Hint:* You do not need to explicitly check every pair for orthogonality, you can argue more generally.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ -3 \\ 0 \end{bmatrix} \qquad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

32. Project $\mathbf{b}$ onto $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} \qquad \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

33. Show that any 2×2 rotation matrix R_θ is an orthogonal matrix.

34. Show that any 3×3 rotation matrix R_x^θ, R_y^θ, R_z^θ (about the x-axis, y-axis, and z-axis, respectively) is an orthogonal matrix.

35. Determine the matrix that implements orthogonal projection onto $\text{span}\{[2 \;\; 3 \;\; 1]^T\}$.

36. Determine a matrix (in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$) that implements orthogonal projection onto $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.

37. Show that for any nonzero vector $\mathbf{v} \in \mathbb{R}^n$, the set of vectors orthogonal to $\mathbf{v}$ form a subspace of $\mathbb{R}^n$.

38. Determine a basis for the set of vectors orthogonal to the following vector.

$$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

39. Determine a basis for the set of vectors orthogonal to every vector in the solution set of the following linear equation.

$$2x_1 + 3x_2 - 4x_3 + 5x_4 = 0.$$

40. Consider the following pair of vectors.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

For this problem, we will be using theorem that isn't too difficult to show: every vector in $\operatorname{Nul} A^T$ is orthogonal to every vector in $\operatorname{Col} A$. Use this to determine two linearly independent vector with integer entries that are orthogonal to every vector in $\operatorname{span}\{\mathbf{u}, \mathbf{v}\}$.

41. For this problem, we will use another theorem that's more difficult to prove: $\operatorname{rank} A + \dim(\operatorname{Nul} A^T) = m$ for any matrix $A \in \mathbb{R}^{m \times n}$. Use this to show that $\operatorname{rank} A = \operatorname{rank} A^T$.

42. Consider the following vectors.[1]

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 9 \\ 0 \\ -11 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -5 \end{bmatrix}$$

   (a) Find the component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ (that is, find the vector $\mathbf{z}$ such that $\mathbf{v}_2 = \hat{\mathbf{v}}_2 + \mathbf{z}$ where $\hat{\mathbf{v}}_2$ is the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$). We will refer to this as $\mathbf{v}_2'$ below.

   (b) Find the component of $\mathbf{v}_3$ orthogonal to $\mathbf{v}_1$. We will refer to this as $\mathbf{v}_3'$ below.

   (c) Find the component of $\mathbf{v}_3'$ orthogonal to $\mathbf{v}_2'$. We will refer to this as $\mathbf{v}_3''$ below.

   (d) Demonstrate that $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3''\}$ is an orthogonal set.

   (e) Find $[\mathbf{u}]_{\mathcal{B}}$ where $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3''\}$ without any row reductions.

43. Repeat the previous problem with the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -5 \end{bmatrix}$$

[1] This problem goes over the *Gram-Schmidt process*, an algorithm for converting a basis into an orthogonal basis.

44.

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 1 & 1 & 3 \\ 8 & 14 & -6 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 & -2 & 9 & 0 \\ 0 & 2 & 2 & -6 & 1 \\ 3 & -5 & 1 & 9 & 5 \\ 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

(a) 🖩 Determine approximately the matrix that implements orthogonal projection onto $\operatorname{Col} A$.

(b) 🖩 Determine approximately the matrix which implements orthogonal projection onto $\operatorname{Col} B$. *Hint:* First determine $\operatorname{rank} B$.

(c) Compare the matrices from the previous two parts. Explain what this implies about the relationship between $\operatorname{Col} A$ and $\operatorname{Col} B$. Justify your answer.

45. Consider the following linear equation

$$x_1 - x_2 + x_3 = 0$$

and vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

(a) Determine a vector $\mathbf{z}$ which is orthogonal to the plane given by the above linear equation.

(b) Determine a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for the plane given by the above linear equation.

(c) Determine a solution to the vector equation $y_1\mathbf{z} + y_2\mathbf{b}_1 + y_3\mathbf{b}_2 = \mathbf{v}$.

(d) Using the previous part, determine the orthogonal projection of $\mathbf{v}v$ onto the plane given by the above linear equation. Justify your answer.

Challenge Problems

46. Show that the row vectors of an orthogonal matrix form an orthonormal set.

47. Show that the product of two orthogonal matrices is orthogonal.

48. Show that $\mathbf{v} \in \operatorname{Nul} A^T$ if and only if $\mathbf{v}$ is orthogonal to every vector in $\operatorname{Col} A$ for any matrix $A \in \mathbb{R}^{m\times n}$ and vector $\mathbf{v} \in \mathbb{R}^m$.

49. Show that $\operatorname{rank} A + \dim(\operatorname{Nul} A^T) = m$ for any matrix $A \in \mathbb{R}^{m\times n}$.

Chapter 13

Least Squares and Linear Models

Basic Exercises

Solve for all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$ using the normal equations.

1.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 3 & -6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -3 \\ -1 & -2 & 7 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -7 \\ -10 \\ 2 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} -1 & 1 \\ -3 & 1 \\ -3 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 11 \\ 15 \end{bmatrix}$$

5.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 7 \\ 2 \\ 3 \end{bmatrix}$$

6. Determine the least squares linear fit to the following data points. Sketch the graphs to verify your answer. You should set up the required matrix equations by hand, but you can use a

computer to determine the coefficients of your model.

$$\{(-5,2),(-2,1),(0,0),(2,4),(6,6)\}$$

7. Determine the least squares quadratic fit to the data points from the previous problem. Sketch the graphs to verify your answer. You should set up the required matrix equations by hand, but you can use a computer to determine the coefficients of your model.

8. Determine the design matrix for the following function model using the data points from the previous problem.

$$f(x) = \beta_1 2^x + \beta_2 x\sin(x) + \beta_3 x^2 + 7\beta_4$$

9. Determine the least-squares fit model for the given data. Round the parameters to the nearest hundredth.

$$\{(-2,0),(-1,5),(0,13),(1,9),(2,5),(3,0)\}$$

10. Determine the design matrix for the following function model using the data points from the previous problem.

$$f(x) = \beta_1 \cos x + \beta_2 \sin x + \beta_3 x + \beta_4$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

11. Given any matrix equation $A\mathbf{x} = \mathbf{b}$, there always exists a least-squares solution.

12. Given any matrix equation $A\mathbf{x} = \mathbf{b}$, there always exists a unique least-squares solution.

13. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} A$.

14. Given a model function $f(x) = \beta_0 + \beta_1^2 x$, we may minimize for least squares error by using the least squares solution to a matrix equation.

15. If X denotes the design matrix for a least squares regression problem, then $X^T X$ is invertible.

16. There are orthogonally diagonalizable matrices that are not symmetric.

17. For any square matrix A, if $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution for each choice of $\mathbf{b}$, then A is invertible.

18. Every $n \times n$ symmetric matrix must has n distinct eigenvalues.

19. For a symmetric matrix, the dimension of each eigenspace is equal to the algebraic multiplicity of the corresponding eigenvalue.

20. For any matrix $A \in \mathbb{R}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$, if $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, then $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

More Difficult Problems

21. Determine a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ given that the columns of A are orthonormal. Furthermore, explain why this solution is unique.

22. Let C be an arbitrary $m \times n$ matrix of rank k and let $\mathbf{b}$ be an arbitrary vector in $\mathbb{R}^m$. Determine the maximum size of a linearly independent set of least squares solutions for $C\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$. Justify your answer. Your solution should be given in terms of m, n, and k.

23. Consider the following function model:

$$f(x) = \beta_0 + \beta_1 x^3 + \beta_2 (2^x)$$

where β_0, β_1, and β_2 are parameters. Determine the design matrix which can be used to compute the parameters that minimize least square error for the following data points.

$$\{(-1, 1), (0, -1), (1, 1), (2, 3), (4, 5)\}$$

24. Consider the following multivariate function model:

$$f(x, y) = \beta_0 + \beta_1 x + \beta_2 y$$

where β_0, β_1, and β_2 are paramters. Computer the paramters minimize least squares error for the following data points.

$$\{(-2, 0, 0), (0, 0, 3), (0, 2, 0), (-2, -3, 3), (1, 5, -1)\}.$$

Justify your answer.

Challenge Problems

25. Suppose we are given four aribtrary data points:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$$

(a) Construct the design matrix for the given data which can be used to find the best-fit curve of the form

$$f_{\beta_1, \beta_2}(\theta) = \beta_1 \cos\theta + \beta_2 \sin\theta$$

where β_1 and β_2 are parameters.

(b) Consider fitting the data with a curve of the form

$$g_{\alpha_1, \alpha_2}(\theta) = \alpha_1 \cos(\theta + \alpha_2)$$

where α_1 and α_2 are parameters. Note that g_{α_1, α_2} is not linear in its parameters. Given $\hat{\alpha_1}$, $\hat{\alpha_2}$, $\hat{\beta}_1$ and $\hat{\beta}_2$—the parameters for the best-fit curves —show that

$$\sum_{i=1}^{4} \left(\hat{\beta}_1 \cos(x_i) + \hat{\beta}_2 \sin(x_i) - y_i\right)^2 \leq \sum_{i=1}^{4} \left(\hat{\alpha_1} \cos(x_i + \hat{\alpha_2}) - y_i\right)^2$$

using the trigonometric identity

$$\cos(a + b) = \cos(a)\sin(b) - \sin(a)\cos(b)$$

In other words, show that the best-fit curve from part (a) has error at least as small as the error of the best-fit curve from part (b).

(c) Compute $\hat{\alpha_1}$ and $\hat{\alpha_2}$ from $\hat{\beta}_1$ and $\hat{\beta}_2$. This implies that, in fact, the errors are equal, and that we can fit to the function g_{α_1,α_2} using the least-squares method.

Chapter 14

Quadratic Forms and Singular Value Decomposition

Basic Exercises

Determine the symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

1.
$$Q([x_1\ x_2\ x_3]^T) = 3x_1^2 + 4x_1x_2 + 5x_2x_1 - x_1x_3$$

2.
$$Q([x_1\ x_2\ x_3]^T) = 3x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + 2x_1x_3 + 2x_2x_3$$

Determine the quadratic form $Q(\mathbf{x})$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

3.
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

Determine the type of the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

4.
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

Determine the singular values of the following matrices

5.
$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 3 & 1 & 1 & -2 \end{bmatrix}$$

Determine a singular value decomposition for each of the following matrices.

6.

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

True/False

Determine if each statement is **true** or **false** and justify your answer. In particular, if the statement is false, provide a counterexample if possible.

7. Every positive-definite symmetric matrix is invertible.
8. For any matrix A and quadratic form $Q(\mathbf{x})$, if $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, then A is symmetric.
9. For any quadratic form $Q(\mathbf{x})$, the vector $\operatorname{argmax}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$ is unique.
10. $\|\mathbf{x}\|^2$ is a quadratic form.
11. A positive definite quadratic form $Q(\mathbf{x})$ satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
12. $\mathbf{x}^T A\mathbf{x}$ defines a quadratic form only if A is symmetric.
13. An orthogonal diagonalization of a symmetric matrix $A = PDP^T$ is also a singular value decomposition of A.

More Difficult Problems

14. Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where A is defined as follows.

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

Determine the following.

(a) $\min_{\|\mathbf{x}\|=1} Q(\mathbf{x})$

(b) $\max_{\|\mathbf{x}\|=1} Q(\mathbf{x})$

(c) $\operatorname{argmin}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$

(d) $\operatorname{argmax}_{\|\mathbf{x}\|=1} Q(\mathbf{x})$

Hint: 4 is an eigenvalue of A.

15. Repeat the previous problem with the following quadratic form.

$$Q([x_1\ x_2\ x_3]^T) = 3x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + 2x_1x_3 + 2x_2x_3$$

16. Consider the following matrices and vector.

$$A = \begin{bmatrix} 2 & -4 & 4 \\ 2 & 2 & 1 \\ -2 & 4 & -4 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 36 & 0 & -36 \\ 0 & 9 & 0 \\ -36 & 0 & 36 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 18 \\ -14 \end{bmatrix}$$

(a) Determine a *reduced* singular value decomposition of A, given that $6\sqrt{2}$ and 3 are its nonzero singular values and $\operatorname{rank}(A) = 2$.

(b) Determine the length of the *shortest* least-squares solution to the equation $A\mathbf{x} = \mathbf{b}$. *Hint:* Use the pseudoinverse of A.

17. Consider the following matrices.

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{bmatrix} \qquad V = \begin{bmatrix} \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{6}/3 \end{bmatrix}$$

(a) Without first determining A^TA or AA^T, determine left singular vectors of A associated with nonzero singular values, i.e., determine the columns of U in the *reduced* SVD of A, where $U\Sigma V^T$ is the reduced SVD of A.

(b) Without first determine A^TA or AA^T, determine the singular values of A.

Challenge Problems

18. let A be as in Problem 17 above. Determine the matrix that implements orthogonal projection onto $\operatorname{Col} A$.